A

CONCISE SYSTEM

OF

MATHEMATICS,

IN THEORY AND PRACTICE,

FOR THE

Use of Schools, Private Students, and Practical Men:

COMPREHENDING

ALGEBRA, PRACTICAL GEOMETRY, LOGARITHMS, PLANE AND SPHERICAL TRIGONOMETRY, MENSURATION OF SURFACES, SOLIDS, HEIGHTS, AND DISTANCES; LAND-SURVEYING, GAUGING, MENSURATION OF ARTIFICERS' WORKS, &c.

WITH

A COPIOUS APPENDIX,

CONTAINING

THE MORE USEFUL PROPOSITIONS OF GEOMETRY, CONIC SECTIONS, FLUXIONS, AND DEMONSTRATIONS OF THE RULES IN THE BODY OF THE WORK.

THE SECOND EDITION,

THOROUGHLY REVISED, WITH MANY IMPORTANT ADDITIONS AND IMPROVEMENTS; BESIDES AN ACCURATE SET OF STEREOTYPED TABLES, COMPRISING LOGARITHMS OF NUMBERS, LOGARITHMIC SINES AND TANGENTS, NATURAL SINES AND TANGENTS, AND THE AREAS OF CIRCULAR SEGMENTS.

ILLUSTRATED BY UPWARDS OF THREE HUNDRED WOOD-CUTS.

BY ALEXANDER INGRAM,

Author of Elements of Euclid, Principles of Arithmetic, Editor of an improved Edition of Melrose's Arithmetic, &c. &c.

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5. The LIMITS of RATIOS, FLUXIONS, and FLUENTS, *previously* forming an Appendix to the Algebra, are now incorporated with the General Appendix, which is so arranged as to exhibit a comprehensive and satisfactory view of the whole theory. And as an introduction to the study of NAVIGATION and NAUTICAL ASTRONOMY, a section on SPHERICAL TRIGONOMETRY, with examples of its application, has been inserted. Hence it will appear that no part of the science really valuable has been omitted.

6. To adapt the work to *all the purposes of teaching*, due regard has also been paid to the variety of Exercises added to each Problem, which will be found more than double the number contained in the first impression,—an addition of the highest importance in a text-book.

7. Besides many new and useful Tables interspersed throughout the work, there are now added TABLES of the LOGARITHMS of NUMBERS from 1 to 10,000, of LOGARITHMIC SINES and TANGENTS to every Degree and Minute, and of NATURAL SINES and TANGENTS to every Five Minutes of the Quadrant. These have all been carefully stereotyped from new types, and, in order to obtain the utmost possible accuracy, rigidly collated with the Tables of BRIGGS, VLACQ, SHERWIN, GARDINER, CALLET, TAYLOR, HUTTON, BABBAGE, BORDA, and also with those of GALBRAITH, which are especially distinguished for accuracy. The Table of the AREAS of CIRCULAR SEGMENTS has likewise been collated with several other more extensive ones.

Such is a brief and cursory view of the leading features now introduced into this edition. But, exclusive altogether of the great amount of new matter, and independent of many minor improvements, the whole work has undergone a careful, rigorous, and minute revision;—what was obscure has been illustrated, and what was defective has been supplied. The errors which had formerly escaped notice have been corrected; and, with the view of securing perfect accuracy, the Author availed himself of the assistance of an eminent Mathematician in examining every calculation; and although it would be presumptuous to assert that the work is immaculate, yet the Publishers feel assured that no error of importance will be found.

Finally, when the Publishers consider the success attending the work in a less perfect shape, they confidently hope that the variety and importance of the contents of the present edition, as well as the perspicuous and familiar manner in which these are treated, taken along with the numerous and extensive additions and improvements introduced throughout, will give it a still higher claim to public favour, and render it more instrumental in facilitating the acquirement of mathematical knowledge, and in disseminating a taste for that science among all classes of students; and, as an additional recommendation, they may venture to affirm, that while it is in many respects the *most complete*, it is unquestionably the *cheapest* work of the kind ever published.

EDINBURGH, *January* 22, 1830.

ORIGINAL PREFACE.

Several treatises on Mensuration have made their appearance within the last fifty years. Among these, Dr Hutton's large work has deservedly acquired the highest celebrity. It treats fully both of the theory and practice of the science, and may be consulted with advantage by persons employed in any kind of measurement. But the scientific part of that work can be read by such only as are well acquainted with the higher branches of Mathematics, and hence the student must have frequent recourse to other publications, to enable him to understand it; while the practical part involves such a multiplicity of rules for the same thing, without distinguishing sufficiently the various cases in which they can be applied, that he is liable to be perplexed with their variety; and nothing has been done by later writers to remove the difficulty.

A book on Mensuration is therefore still wanted, embracing the *whole* theory and practice in such a way, that both, though kept separate, may be rendered intelligible to every reader, without the necessity of having recourse to other publications, and arranged in a condensed form, so as to comprise a complete system of the science in a small compass. Such are the objects of the present publication.

The practical part of this work consists of plain rules for performing the various operations requisite in *Trigonometry, Mensuration, Surveying, Gauging, &c.* These rules are illustrated by proper examples, one or more of which is wrought for the assistance of the learner. A demonstration of the rule is sometimes annexed to it in the form of a note, when this can be done in an easy and concise manner. But the more difficult demonstrations are reserved for the Appendix.

By pursuing this method, the author has endeavoured to render the book fit for the use of every person who wishes to study Mensuration with facility and success. The treatise on Practical Geometry, which is prefixed to the Trigonometry, will enable the student to draw his figures; while the rules delivered in the following part of the work will direct him how to find their contents, and the lengths of their lines; and a little reflection will qualify him to compare these lengths or contents with one another. In such a state, the work will be found a most useful guide to practical measurers, and well adapted to the use of schools. The rules may be applied directly in all ordinary cases. If one shall occur which requires investigation, the method of conducting this process may be learned from the treatise on Algebra, which is prefixed to the work.

In the treatise on Algebra, great care has been taken to remove irregularities, and other difficulties, of which beginners usually complain; and the demonstrations of the fundamental rules are generalized, and deduced from one principle intimately connected with the nature of abstract quantity. A short Appendix is annexed to this part, which treats of the management of indeterminate problems, of the relations of variable quantities, and of the limits of ratios, with as much of the practice of Fluxions and Fluents as is requisite in this performance.

The Practical Geometry, though short, contains every thing necessary for what follows. Some new methods of operation are introduced, and the lines and angles are generally expressed in numbers.

In the Mensuration, the application of the series for finding the circumference of the circle, of which the diameter is unit, has been taken from Euler, and appears to be as simple as it can be made. New rules are given for approximating to the length of an arc of a circle, and to the area of a segment of it, which are both easier and more accurate than those formerly employed by the use of roots. The method of forming the most common solids with pasteboard is introduced, because it renders the reader familiar with their shapes, and illustrates the rules for finding their superficies.

Land-surveying, Gauging, &c., are the application of Trigonometry and Mensuration to practical purposes. Great plainness has therefore been studied in explaining them, and the shortest, easiest, and most approved methods of practice have been adopted.

The Appendix is appropriated to the demonstration of the rules delivered in the preceding parts of the work. Such of the principles of Geometry and of Conic Sections are introduced as are necessary for enabling the reader to understand the demonstration of the rules, without having recourse to other publications. Here accuracy is rigidly adhered to. Many new demonstrations are given, which are more simple than those that were formerly employed. The theory of Parallel Lines has been rendered as plain and concise as possible. The principles of Conic Sections have been deduced from the ratio of the curve, or its relation to the focus and directrix,—a method which has been generally held by mathematicians to be superior to every other. The leading propositions only are delivered; but they are so regulated as to introduce principles from which the other properties of these curves may be easily derived.

The student who has abundance of time should begin with Algebra, and then read the Appendix to the work and the Practical Geometry together; after which, he should go regularly through the book, in the order in which it is printed. In doing this, he may acquire as much knowledge of Mathematics as will be sufficient for ordinary purposes, and be enabled to prosecute that most extensive science with pleasure and advantage. If his time and other pursuits do not admit of such a regular progress, he may study separately any of the practical branches best adapted to his taste, or the purpose to which he intends to apply them.

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NOTE.

In the formula for measuring heights by the barometer, page 189, the number ·00245 should, according to General Roy's experiments, be ·00244. Laplace makes it ·00222, and others ·00223. In the table, page 280, the value of C for Teak should be 15550. In line 17, page 281, the number 32 should, according to Mr Bevan, be 16; but Mr Barlow says that it is 32 in the usual methods of fixing beams in ordinary erections.

ALGEBRA.

DEFINITIONS.

ALGEBRA is a general method of computation and of investigation, in which quantities are represented by letters, and their relations pointed out by characters.

CHARACTERS EXPLAINED.

1. $+$ *plus*, is the sign of addition, as $a+b$ signifies the quantity represented by b added to that represented by a.

2. $-$ *minus*, is the sign of subtraction, as $a-b$ denotes the quantity b taken from the quantity a.

3. $\times$ *into*, is the sign of multiplication, as $a\times b$ represents the product of a by b, or of b by a. Instead of this sign we often use a point, or write the letters together as in one word: thus $a{\cdot}b$ or ab signifies $a\times b$.

4. $\div$ *by*, is the sign of division, but it is generally expressed by placing the dividend above the line and the divisor below it, in the form of a fraction: thus $a\div b$ or $\frac{a}{b}$ signifies a divided by b.

5. $: :: :$ is the sign of proportion, as $a:b::c:d$ is read, As a is to b, so is c to d.

6. $=$ *equal to*, is the sign of equality: thus $a=b$ signifies a is equal to b.

7. $>$ $<$ are signs of greater and less: thus $a>b$, a is greater than b; $a<b$, a is less than b.

8. $7a$. A number prefixed to a letter is called its *coefficient*, and shews how often the letter is to be taken; as here, 7 times a.

9. $(a+b)\times c$. A parenthesis enclosing letters, or a line drawn over them, is called a *vinculum*, and points out how many are to be multiplied, divided, &c.; as here, the sum of a and b is to be multiplied by c.

10. aaa. When the same letter is repeated twice, or oftener, it is understood to be multiplied as often into itself, and the product is called a power of the quantity represented by that letter: thus aa is the second power or square of a, aaa is the third power of a, &c.; and in relation to these powers the quantity is called the first power of itself.

11. a^3. Instead of repeating the same letter, we generally place a figure *above it towards the right hand*, to shew how

often it is repeated; as a^3 is the third power of a, a^4 the fourth power, a^n the power of a denominated by the number n.

12. The character placed above is called the exponent of the power.

13. A quantity which, multiplied by itself, produces another quantity, is called the root of that other, and, in numbers, is generally denoted by the mark $\sqrt{}$: thus $\sqrt{9}$ is the square root of 9, $\sqrt[3]{8}$ is the cube root of 8, $\sqrt[4]{81}$ is the fourth root of 81.

14. In letters, a fractional exponent is generally used to express the root, and then the upper figure denotes the power, and the under figure the root: thus $a^{\frac{2}{3}}$ is the third root of the second power of a, $a^{\frac{1}{4}}$ is the fourth root of the first power of a, or of a itself.

Quantities which have the sign $+$ before them are said to be positive or affirmative, and those which have the sign $-$ negative.

The following examples will illustrate these characters, and shew their use, in which any values may be affixed to the letters. Suppose $a=12$, $b=3$, $c=2$, $d=4$, $e=5$, $f=9$, $g=25$, $h=7$, $i=11$, $k=1$.

1.	$a+b-c+d$	$=$	17
2.	$4a-5b+4c-7d$	$=$	13
3.	$ab-2cd+4be-3cf$	$=$	26
4.	$8aa-5ab+10ac-4bc+4bb$	$=$	1224
5.	$6a^3-4a^2b+2ab^2-7b^3$	$=$	8667
6.	$2a^2bc+3ab^2c-5abc^2$	$=$	1656
7.	$(a-b)\times 2c-d\times(b+c)$	$=$	16
8.	$(a+b)\times(g-h)\times(i-k)$	$=$	2700
9.	$2a^2c-\frac{a^2}{c}+\frac{a}{c^2}$	$=$	507
10.	$\frac{3a}{c}+\frac{4g}{5d}-\frac{6d}{b}$	$=$	15
11.	$\frac{a+b+e}{d+k}+\frac{g}{e}$	$=$	9
12.	$\frac{3a}{b}\times\frac{2c}{d}\times\frac{4g}{d}$	$=$	300
13.	$\frac{ac}{d}+\frac{gf}{c}-\overline{cd}^{\frac{1}{3}}$	$=$	$116\frac{1}{2}$
14.	$\frac{\sqrt{ae+d}}{i-h}$	$=$	2
15.	$(g+c)^{\frac{1}{3}}+(ai-b^2-c)^{\frac{1}{2}}$	$=$	14

16. $aa^{\frac{1}{2}}b^{\frac{1}{2}}+\overline{bf}^{\frac{1}{3}}-\overline{df}^{\frac{1}{2}}$ . . = 69

17. $aa^{\frac{1}{2}}b^{\frac{1}{2}}\times\overline{bf}^{\frac{1}{3}}\times\overline{df}^{\frac{1}{2}}$ . . . . = 1296

18. $d^2-d+d^{\frac{1}{2}}-(d+e)^{\frac{1}{2}}$ . . . = 11

19. $\frac{(def+ef)^{\frac{1}{2}}}{bcd+k}+\left(\frac{def+ef+g}{f+k}\right)^{\frac{1}{2}}$ . . = $5\frac{3}{8}$

20. $\left(\frac{a^2}{f^{\frac{1}{2}}}+\frac{(hi+d)^{\frac{1}{2}}}{(g+c)^{\frac{1}{3}}}+\frac{d^4}{f-k}-(b+e)^{\frac{1}{3}}\right)^{\frac{1}{2}}$. = 9

21. $\frac{4a\times\overline{ab}^{\frac{1}{2}}}{2c+(c+2)^{\frac{1}{2}}}+\frac{a^2}{b}\times\left[h+\left(\frac{a^3}{2}-bef\right)^{\frac{1}{3}}\right]=\frac{48\sqrt{36}}{4+\sqrt{4}}+\frac{144}{3}$

$\times\left[7+\left(\frac{1728}{2}-135\right)^{\frac{1}{3}}\right]$ . . = 816

22. $3d^{\frac{1}{2}}+2a\times(2a+b-c)^{\frac{1}{2}}$ . . = 126

23. $3ab+(bc+(3ab-2c^2)^{\frac{1}{2}})^{\frac{1}{2}}$ . . = 112

24. $\frac{(8ci+(2a-d)^2)^{\frac{1}{2}}-(2a-d)}{2c}$ . . = 1

NOTE.—All the fundamental operations of algebra depend upon this single principle, viz. When a quantity is to be increased or diminished by other quantities, the same result will be obtained in whatever order the procedure is carried on, provided none of the quantities be neglected. This is manifest from the nature of quantity, which has no relation to order. Thus, if we have to add 7 and 5, and to subtract 3, we may first subtract 3 from 7, and add the remainder to 5; or we may subtract 3 fróm 5, and add the remainder to 7; or we may add 7 to 5, and from the sum 12 subtract 3: the result in every case is 9. Again, if we have to multiply 12 and 6, and to divide by 3; we may first divide 12 by 3, and multiply the quotient by 6; or we may divide 6 by 3, and multiply the quotient by 12; or we may multiply 12 by 6, and divide the product by 3: the result in every case is 24.

ADDITION.

CASE 1.—WHEN the quantities are alike; if the signs be the same, add the coefficients, but if different, subtract them, and to the sum or difference prefix the sign of the greater, and annex the common letter or letters.

CASE 2.—When the quantities are unlike; write them one after another, with their proper signs and coefficients.

NOTE 1.—When there are more than two like quantities, add the coefficients of those which have + into one sum, and of those which have — into another, and subtract the less sum

from the greater. The arrangement of the quantities is arbitrary, and must often be altered to bring like quantities under like.

NOTE 2.—A quantity which has no sign prefixed is understood to have +, and a quantity which has no coefficient or exponent is supposed to have 1.

1. $$\begin{array}{l} 3a-5b+4c-3d-2e \\ \underline{6a+2b-7c-4d+8e.} \\ 9a-3b-3c-7d+6e. \end{array}$$

2. $$\begin{array}{l} 8a^2b-5ab^2-8abc+4bc^2 \\ -2a^2b+6ab^2-abc-4bc^2. \end{array}$$

3. $$\begin{array}{l} 6ab+2ac-3bc+4bd \\ -7ab-3ac+6bc+5bd. \end{array}$$

4. $$\begin{array}{l} 8a^{\frac{1}{2}}b^3-7a^2bc^{\frac{1}{2}}-4ab^{\frac{1}{2}}c^2+3abc \\ 7a^{\frac{1}{2}}b^3+7a^2bc^{\frac{1}{2}}-3ab^{\frac{1}{2}}c^2-4abc. \end{array}$$

5. $$\begin{array}{l} 8a^3b-7a^2b^2+4ab^3-a^4+b^4 \\ 7a^2b^2-8ab^3+4a^4-3b^4-2a^3b \\ 6ab^3-2a^4+3b^4-7a^3b+5a^2b^2 \\ 5a^4-7b^4-6a^3b+5a^2b^2-3ab^3 \\ \underline{7b^4-2a^3b+2a^2b^2-ab^3+4a^4.} \\ 10a^4-9a^3b+12a^2b^2-2ab^3+b^4. \end{array}$$

6. $$\begin{array}{l} a+(a-v)^{\frac{1}{2}}+5 \\ 2a+(a-v)^{\frac{1}{2}}-10. \end{array}$$

7. $$\begin{array}{l} a+(a+v)^{\frac{1}{2}}+5 \\ 2a+(a-v)^{\frac{1}{2}}-10. \end{array}$$

8. $$\begin{array}{l} a^3+a^2-a \\ a^{\frac{3}{2}}+a^{\frac{2}{3}}-a^{\frac{1}{2}} \\ a^{\frac{3}{2}}+a^2-a^{\frac{1}{3}}. \end{array}$$

9. $$\begin{array}{l} 10(a+e)^{\frac{1}{2}}+(a-e)^{\frac{1}{2}} \\ -(a+e)^{\frac{1}{2}}-(a-e)^{\frac{1}{2}}. \end{array}$$

10. $$\begin{array}{l} a^3+3a^2+5+(a-v)^{\frac{1}{2}}+a+6(a+v)^{\frac{1}{2}} \\ 3a^2-2a+6a^3-2(a-v)^{\frac{1}{2}}+10-6(a+v)^{\frac{1}{2}} \\ 7a-5a^3-2a^2+4(a+v)^{\frac{1}{2}}-b+8(a-v)^{\frac{1}{2}} \\ 8c-6a^2+4a^3-2(a-v)^{\frac{1}{2}}+7-6a \\ \underline{7a^2-8a^3+4-5(a+v)^{\frac{1}{2}}+3a-8(a+v)^{\frac{1}{2}}} \\ 5a^2-2a^3+26+5(a-v)^{\frac{1}{2}}+3a-9(a+v)^{\frac{1}{2}}-b+8c. \end{array}$$

NOTE.—If the difference $a-b$ is to be added to $3a$, we may first subtract b from a, and then add the remainder to $3a$; or we may subtract b from $3a$, and add a to the remainder. Here we first add a to $3a$, and then subtract b, and it becomes $4a-b$. If $2a+b$ is to be added to $3a-4b$, we add $2a+b$ to $3a$, and it becomes $5a+b$; from which we take $4b$, and it becomes $5a-3b$.

SUBTRACTION.

Rule.—Change the signs of the subtrahend from + to —, or from — to +, and then proceed as in Addition.

1. From $8ab - 2cd + 5ac - 7ad$
   Take $3ab + 4cd + 5ac - 2ad.$
   $5ab - 6cd \quad * \quad - 5ad.$

2. From $18a^2b - 12abc - 3ab^2 + b^3$
   Take $6a^2b + 3abc - 4ab^2 - 3b^3.$

3. From $a^2x^2c - 5ax^2c^2 + 2a^2xc^2$
   Take $3a^2x^2c + 4ax^2c^2 + 2a^2xc^2.$

4. From $-3a^3b^{\frac{1}{2}} + 2a^2bc^{\frac{1}{2}} - 5a^{\frac{1}{2}}b^2c$
   Take $4a^3b^{\frac{1}{2}} - 2a^2bc^{\frac{1}{2}} - 5a^{\frac{1}{2}}b^2c.$

5. From $3bd + 2a$
   Take $2bd - 3a - b.$

6. From $\frac{(a-b+2)^{\frac{1}{2}}}{a+b}$
   Take $-\left(\frac{a-b+2}{a+b}\right)^{\frac{1}{2}}.$

7. From $2bc - 11a - d$
   Take $d + 11a - 2bc.$

8. From $a^3 + a^{\frac{3}{2}}$
   Take $a^3 - a^{\frac{3}{2}}.$

Note.—If we are to subtract $a - c$ from $3a$, we may first subtract c from a, and then subtract the remainder from $3a$; or we may add c to $3a$, and then subtract a from the sum. Here we subtract the whole a from $3a$, and add c to the remainder. If $a - c$ is to be subtracted from $3a + 2c$, we subtract a as before from $3a$, and then add c, and the remainder becomes $2a + 3c$. Now all this is performed by changing the signs of the quantity $a - c$ into $-a + c$, and then adding it.

These considerations lead us to perceive how we may add or subtract any two terms, without regard to the other terms with which they are connected.

MULTIPLICATION.

Multiply the coefficients, and to the product annex the letters of both factors.

If the sign of the multiplier is +, make the sign of the product the same with that of the multiplicand. If the sign of the multiplier is —, make the sign of the product contrary to that of the multiplicand.

Hence, like signs produce +, and unlike signs —.

If the multiplicand is compound, multiply each term of it separately by the multiplier.

If the multiplier is compound, multiply first by one of its terms, then by another, &c. and afterwards add the products.

Powers of the same quantity are multiplied by adding their exponents.

1. Multiply $5a-4b+3c-2d+e-1$
by $5a.$

$$\overline{25a^2-20ab+15ac-10ad+5ae-5a.}$$

2. Multiply $6aa-7ab+4ac-b^2+2bc-c^2$ by $4ab$.
3. ...... $3a-2b$ by $-2a+4b$.
4. ...... $5a^2-3ab+4b^2$ by $6a-5b$.
5. ...... a^2+ab+b^2 by $a-b$.
6. ...... a^4-x^4 by a^4-x^4.
7. ...... $2x^2-3xy+6$ by $3x^2+3xy-5$.
8. ...... $5a^2-4ax+3x^2$ by $2a^2-3ax-4x^2$.
9. ...... $2a^2x^2-2ax+3a^2$ by $3a^2x^2+4ax-5a^2$.
10. ...... $x^2-ax+\frac{1}{4}a^2$ by $x^2+ax-\frac{1}{4}a^2$.
11. ...... $x-\frac{1}{2}a$ by $x+\frac{1}{2}a$.
12. ...... x^2+xy+y^2 by x^2-xy+y^2.
13. ...... $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
14. ...... $3a-2b+2c$ by $2a-4b+5c$.
15. ...... $a^3-3a^2b+3ab^2-b^3$ by $a^2-2ab+b^2$.
16. ...... a^3-3a^2+3a-1 by a^2-2a+1.*

Since $1\times b+1\times b+1\times b=3\times b$, if as many units be taken as are in a, and each of them be multiplied by b and the products be added, the sum will be $a\times b$: but b taken as many times as there are units in a produces $b\times a$; therefore $a\times b$ is the same with $b\times a$, or $ab=ba$. In like manner abc, acb, bac, bca, cab, cba, are all the same, so that the factors may be placed in any order.

Again, since $ma=a+a+a$, &c. being repeated m times,
and $mb=b+b+b$, &c. being repeated m times;
therefore $ma+mb=(a+b)+(a+b)+(a+b)$ repeated m times, that is, $ma+mb=m(a+b)$. In like manner $ma-mb=m(a-b)$.

In multiplying $a-b$ by c, we may either first subtract and then multiply, or first multiply and then subtract. The latter is the order in algebra: we first multiply a by c, which makes ac, and then b by c, and it makes bc, and subtract the latter product from the former to get the just product $ac-bc$, where the signs are the same with those of the multiplicand.

In multiplying $a-b$ by $c-d$, we first multiply $a-b$ by c as before, and it produces $ac-bc$; then we multiply $a-b$ by d, and it produces

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* ANSWERS.—(2.) $24a^3b-28a^2b^2+16a^2bc-4ab^3+8ab^2c-4abc^2$. (3.) $-6a^2+16ab-8b^2$. (4.) $30a^3-43a^2b+39ab^2-20b^3$. (5.) a^3-b^3. (6.) $a^8-2a^4x^4+x^8$. (7.) $6x^4-3x^3y+8x^2-9x^2y^2+33xy-30$. (8.) $10a^4-23a^3x-2a^2x^2+7ax^3-12x^4$. (9.) $6a^4x^4+2a^3x^3-a^4x^2-8a^2x^2+22a^3x-15a^4$. (10.) $x^4-a^2x^2+\frac{1}{2}a^3x-\frac{1}{16}a^4$. (11.) $x^2-\frac{1}{4}a^2$. (12.) $x^4+x^2y^2+y^4$. (13.) $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$. (14.) $6a^2-16ab+19ac+8b^2-18bc+10c^2$. (15.) $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$. (16.) $a^5-5a^4+10a^3-10a^2+5a-1$.

$ad - bd$, which we subtract from the former product, or change its signs, and it becomes $-ad + bd$, where the signs are contrary to those of the multiplicand.

The first and last terms shew that quantities with like signs produce +, and the other two terms shew that those which have unlike signs produce —.

DIVISION.

When the divisor is a simple quantity, write it under the dividend in the form of a fraction, then cancel like quantities in them, and divide the coefficients by their greatest common measure.

When the signs are alike, the sign of the quotient is +; but if they be unlike, it is —.*

Powers of the same quantity are divided by subtracting the exponent of the divisor from that of the dividend; the remainder is the exponent of the quotient.

If the dividend be compound, divide each term of it separately by the divisor.

Divide the following:

1. $56a^2b^3c$ by $8ab^3$. . . Ans. $7ac$.
2. $54xy^2$ by $36x^2y$. . . $3y \div 2x$.
3. $63a^3b^2c^3 - 42a^2b^3c^3$ by $14a^2b^2c^2$. $4\frac{1}{2}ac - 3bc$.
4. $24x^3y - 18x^2y^2 + 15xy^3$ by $30xy^2$. $4x^2 \div 5y - \frac{3}{5}x + \frac{1}{2}y$.

When the divisor is compound, arrange the terms of the dividend and divisor according to the powers of the same letter. Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient, then multiply the whole divisor by this term, and subtract the product from the dividend; the remainder is a new dividual, with which proceed as before.

Note.—When the remainder is a simple quantity, place the divisor below it in the form of a fraction, and annex it with its proper sign to the quotient.

5. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$

$$
\begin{array}{l}
a-b)a^3 - 3a^2b + 3ab^2 - b^3(a^2 - 2ab + b^2 \\
\qquad\quad \underline{a^3 - a^2b} \\
\qquad\qquad\;\, -2a^2b + 3ab^2 \\
\qquad\qquad\;\, \underline{-2a^2b + 2ab^2} \\
\qquad\qquad\qquad\qquad +ab^2 - b^3 \\
\qquad\qquad\qquad\qquad +ab^2 - b^3.
\end{array}
$$

* This is evident; for the divisor multiplied by the quotient must produce the dividend with its proper sign. The whole operation depends upon this principle, that the value of a quantity is not altered by both multiplying and dividing it by the same quantity.

6. Divide $8a^3-4a^2b-6ab^2+3b^3$ by $2a-b$

$$\begin{array}{l} 2a-b)8a^3-4a^2b-6ab^2+3b^3(4a^2-3b^2 \\ \quad\quad\; 8a^3-4a^2b \\ \hline \quad\quad\quad\quad\quad\quad -6ab^2+3b^3 \\ \quad\quad\quad\quad\quad\quad -6ab^2+3b^3. \end{array}$$

7. $3b^3+3ab^2-4a^2b-4a^3$ by $a+b$. Ans. $-4a^2+3b^2$.
8. a^4-b^4 by $a-b$. . . $a^3+a^2b+ab^2+b^3$.
9. $8a^4+2a^2b^2-3b^4$ by $2a^2-b^2$. . $4a^2+3b^2$.
10. $2a^2x^2-5ax+2$ by $2ax-1$. . $ax-2$.
11. $x^2-x+\frac{1}{4}$ by $x-\frac{1}{2}$. . . $x-\frac{1}{2}$.
12. $21a^5-21b^5$ by $7a-7b$. . . $3a^4+3a^3b$, &c.
13. $x^4-y^4+2y^2z^2-z^4$ by $x^2+y^2-z^2$. $x^2-y^2+z^2$.
14. $1+a$ by $1-a$. . . . $1+2a+2a^2$, &c.
15. $8x^2-15y^2+23yz-2xy-8xz-6z^2$ by $2x-3y+z$. Ans. $4x+5y-6z$.
16. $a^2-2ab+b^2$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$. $a^{\frac{3}{2}}-ab^{\frac{1}{2}}-a^{\frac{1}{2}}b+b^{\frac{3}{2}}$.
17. $6x^4-96$ by $3x-6$. . $2x^3+4x^2+8x+16$.
18. $1+2x$ by $1-x$. . . $1+\frac{3x}{1-x}$.

FRACTIONS.

A FRACTION is one or more parts of a unit. The denominator expresses the number of parts into which the unit is supposed to be divided, and the numerator expresses the number of these parts of which the fraction consists: thus, in the fraction $\frac{m}{n}$, n denotes the number of parts into which the unit is divided, and m points out the number of these parts of which the fraction consists. If the unit had been divided into $2n$ parts, then the fraction must have consisted of twice the number of these parts, and would have been $\frac{2m}{2n}$. In the same manner it might be expressed by $\frac{3m}{3n}$, $\frac{rm}{rn}$, &c.

Hence, the value of a fraction is not altered by multiplying or dividing both its terms by the same quantity.

REDUCTION.

PROBLEM I.

To reduce an integer to the form of a fraction.

If the denominator be given, multiply the integer by it for

the numerator, and under the product place the denominator. If no denominator is given, place unit for it.

Hence, a mixed quantity may be reduced to the form of a fraction by multiplying the integer by the denominator of the fraction, and adding the numerator to the product for the numerator, below which place the denominator.

1. Reduce $3a$ to a fraction, of which the denominator is $2b$. Ans. $\frac{6ab}{2b}$.

2. Reduce $a+\frac{b}{c}$ to an improper fraction. $\frac{ac+b}{c}$.

3. . . . . $x+\frac{a^2}{x}$. . . . $\frac{x^2+a^2}{x}$.

4. . . . . $x-\frac{a^2x^2}{x}$. . . $\frac{x^2-a^2x^2}{x}$.

5. . . . . $5-\frac{3x}{a}$. . . . $\frac{5a-3x}{a}$.

6. . . . . $a-\frac{ab-a^2}{2b}$.* . . $\frac{ab+a^2}{2b}$.

7. . . . . $a-x-\frac{a^2x^2}{2x}$. . . $\frac{2ax-2x^2-a^2x^2}{2x}$.

8. . . . . $a+1-\frac{x-1}{b}$. . . $\frac{ab+b-x+1}{b}$.

9. . . . . $1+3a-\frac{4x-5}{4x}$. . $\frac{12ax+5}{4x}$.

PROBLEM II.

To reduce an improper fraction to an integer or a mixed quantity.

Divide the numerator by the denominator, the quotient is the integer, the remainder, with the divisor below it, constitutes the fraction.

1. Reduce $\frac{ab+b^2}{a}$ to a mixed quantity. Ans. $b+\frac{b^2}{a}$.

2. . . . . $\frac{ax+2x^2}{a+x}$. . . $x+\frac{x^2}{a+x}$.

3. . . . . $\frac{x^2-y^2}{x+y}$. . . . $x-y$.

4. . . . . $\frac{x^3-y^3}{x-y}$. . . . x^2+xy+y^2.

---

* When a fraction has the sign — before it, all the signs of the numerator are to be changed. Here $ab-a^2$ becomes $-ab+a^2$

5. Reduce $\frac{12x^2-18}{3x}$. . Ans. $4x-\frac{6}{x}$.

6. . . . . $\frac{4x^2-2x}{2x^2-x+1}$. . . . $2-\frac{2}{2x^2-x+1}$.

PROBLEM III.

To reduce fractions of different denominators to others of the same value which have a common denominator.

Multiply each of the numerators into all the denominators, except its own, for the new numerators, and all the denominators together for the common denominator.

1. Reduce $\frac{3a}{b}$ and $\frac{2a}{3c}$ to a common denominator.
Ans. $\frac{9ac}{3bc}$ and $\frac{2ab}{3bc}$.

2. . . . . $\frac{a+b}{c}$ and $\frac{3d}{m}$. . . . $\frac{am+bm}{cm}$ and $\frac{3cd}{cm}$.

3. . . . . $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$. . . $\frac{adn}{bdn}$, $\frac{bcn}{bdn}$, $\frac{bdm}{bdn}$.

4. . . . . $\frac{2a}{3}$, $\frac{3b}{4}$, and $\frac{5c}{3d}$. . . $\frac{8ad}{12d}$, $\frac{9bd}{12d}$, $\frac{20c}{12d}$.

5. . . . . $2a$ and $\frac{3b}{4}$. . . $\frac{8a}{4}$ and $\frac{3b}{4}$.

6. $\frac{7a^2}{x}$, $\frac{a}{4}$, $\frac{a^2-x^2}{a+x}$. . $\frac{28a^3+28a^2x}{4ax+4x^2}$, $\frac{a^2x+ax^2}{4ax+4x^2}$, $\frac{4a^2x-4x^3}{4ax+4x^2}$.

PROBLEM IV.

To reduce a fraction to lower terms.

Divide its numerator and denominator by any quantity which measures both.

1. Reduce $\frac{ax^2-x^3}{ax+x^2}$ to lower terms. . Ans. $\frac{ax-x^2}{a+x}$.

2. . . . . $\frac{6a^2-12x^2}{3a-6x}$. . . . . $\frac{2a^2-4x^2}{a-2x}$.

3. . . . . $\frac{4a^2x^3}{2ax-2a^2}$. . . . . . $\frac{2ax^3}{x-a}$.

4. . . . . $\frac{36a^2x^2}{24a^3x}$. . . . . $\frac{3x}{2a}$.

5. . . . . $\frac{9a^2-12ax+4x^2}{3ax-2x^2}$. . . . $\frac{3a-2x}{x}$.

The greatest divisor of the coefficients is found as in arithmetic, and the greatest simple divisor of the letters is discovered by inspection.

To find the greatest compound divisor.

Divide the greater by the less and the divisor by the remain-

der continually, till nothing remain: the last divisor is the greatest common measure.

Note.—The several divisors must be first divided by the greatest simple quantity which measures all their terms before they are used. Also the dividend must be sometimes multiplied by a simple quantity to make the division succeed. And any compound quantity in a remainder which does not measure the divisor from which it proceeds, may be taken out of it.

What is the greatest common measure of

1. $\frac{a^4-b^4}{a^5+a^3b^2}$. . . Ans. a^2+b^2.

2. $\frac{x^2-y^2}{x^4-y^4}$. . . . x^2-y^2.

3. $\frac{x^4-y^4}{x^3-x^2y-xy^2+y^3}$. . . . x^2-y^2.

4. $\frac{6x^3-6x^2y+2xy^2-2y^3}{12x^2-15xy+3y^2}$. . . $x-y$.

5. $\frac{3bcq+30mp+18bc+5mpq}{24ad-7fgq-42fg+4adq}$.* . $q+6$.

6. $\frac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}$. . $x+2a$.

7. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. . $\frac{x-1}{y}$.

8. . . . . $\frac{ax+x^2}{ac^2+c^2x}$. . . Divide by $a+x$.

9. . . . . $\frac{x^3-a^2x}{x^2+2ax+a^2}$. . . by $x+a$.

10. . . . . $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. . by a^2-x^2.

11. . . . . $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$. . by $a+x$.

12. . . . . $\frac{a^5+a^3b^2}{a^4-b^4}$. . . by a^2+b^2.

ADDITION AND SUBTRACTION.

Reduce the fractions to a common denominator, if they have different ones; then add or subtract their numerators, and

* In fractions like this, where a letter is but of one dimension in either the numerator or the denominator, divide it into two parts, one of which has that letter in every term; then find the common measure of these two parts, and try whether it will divide the other quantity. Here the parts of the denominator are $4adq+24ad$ and $-7fgq-42fg$, and the common measure of these is $q+6$, which succeeds.

under the sum or the remainder write the common denominator, for the sum or the difference of the fractions.

1. Add $\frac{3a}{4}$, $\frac{5a}{6}$, $\frac{a}{3}$ together. . Ans. $\frac{23a}{12}$.

2. . . $\frac{x-3}{4}$, $\frac{5x+2}{3}$, $\frac{7x}{5}$. . . $\frac{199x-5}{60}$.

3. . . . $4x$, $\frac{3x^2}{2a}$, $\frac{x+a}{3x}$. . $\frac{9x^3+24ax^2+2ax+2a^2}{6ax}$.

4. . . $2a+\frac{a+3}{5}$, $4a+\frac{2a-5}{4}$. . $6a+\frac{14a-13}{20}$.

5. . . $\frac{x}{a}-\frac{x}{2a}$, $\frac{3x}{a}-\frac{4x}{2a}$, $4x$. . $4x+\frac{3x}{2a}$.

6. . . $x-\frac{a^2}{x}$, $a-\frac{a-x}{c}$. . $a+x+\frac{x^2-ax-a^2c}{cx}$.

7. From $3a-\frac{4x}{a}$, take $a+\frac{5x}{3a}$. . $2a-\frac{17x}{3a}$.

8. . . . $\frac{7x}{v}-\frac{4x^2}{5v}$, take $\frac{3x}{7v}-\frac{2x^2}{17v}$. . $\frac{46x}{7v}-\frac{58x^2}{85v}$.

9. . . . $\frac{x-y}{2a}$, take $\frac{x+y}{3a}$. . . . $\frac{x-5y}{6a}$.

What is the sum and the difference of

10. $\frac{x+y}{2}$ and $\frac{x-y}{2}$? . . Ans. x and y.

11. $\frac{1}{a-b}$ and $\frac{1}{a+b}$. . . $\frac{2a}{a^2-b^2}$ and $\frac{2b}{a^2-b^2}$.

12. $2x+\frac{3x}{a}$ and $x-\frac{2x-2a}{3c}$.

Sum $3x+\frac{9cx-2ax+2a^2}{3ac}$, and Diff. $x+\frac{9cx+2ax-2a^2}{3ac}$.

MULTIPLICATION AND DIVISION.

Multiply the numerators together for the numerator of the product, and the denominators together for its denominator.

In division, invert the divisor and multiply as before.

1. Multiply $\frac{2x}{3}$ by $\frac{5x}{6}$. . . Ans. $\frac{5x^2}{9}$.

2. . . . . $\frac{x+a}{a+c}$ by $\frac{a}{x}$. . . $\frac{ax+a^2}{ax+cx}$.

3. . . . . $b+\frac{bx}{a}$ by $\frac{a}{x}$. . . . $\frac{ab}{x}+b$.

4. . . . . $\frac{ad}{2bc}$ by $\frac{4c}{d}$. . . $\frac{2a}{b}$.

5. Divide $\frac{x}{3}$ by $\frac{2x}{9}$. . . Ans. $1\frac{1}{2}$.

6. . . . . $\frac{2x^2}{a^2+x^2}$ by $\frac{x}{x+a}$. . . $\frac{2x(x+a)}{a^2+x^2}$.

7. . . . . $\frac{x}{x-1}$ by $\frac{x}{2}$. . . $\frac{2}{x-1}$.

8. . . . . $\frac{x^4-a^4}{x^2-2ax+a^2}$ by $\frac{x^2+ax}{x-a}$. . . $\frac{x^2+a^2}{x}$.

9. . . . . $\frac{a+x}{b^2+2bx+x^2}$ by $\frac{1}{b+x}$. . . $\frac{a+x}{b+x}$.

The four fundamental rules require the aid of those for fractions, when any terms of the given quantities, or of those which arise in the course of the operation, are fractional.

10. Multiply $\frac{a^2}{9}-\frac{ax}{3}+\frac{x^2}{4}$ by $\frac{a}{3}-\frac{x}{2}$. Ans. $\left(\frac{a}{3}-\frac{x}{2}\right)^3$.

11. . . . . $\frac{a}{b}+\frac{c}{d}$ by $\frac{a}{b}-\frac{c}{d}$. . . $\frac{a^2}{b^2}-\frac{c^2}{d^2}$.

12. . . . . $\frac{3a}{4b}+\frac{2c}{3d}$ by $\frac{3a}{4b}-\frac{2c}{3a}$. . $\frac{9a^2}{16b^2}-\frac{4c^2}{9d^2}$.

13. . . . . $\frac{x^2}{a^2}+\frac{xy}{ac}+\frac{y^2}{c^2}$ by $\frac{x}{a}-\frac{y}{c}$. . $\frac{x^3}{a^3}-\frac{y^3}{c^3}$.

14. Divide a^2+b^2 by $a+b$. . . $a-b+\frac{2b^2}{a+b}$.

15. . . . . $\frac{x^2}{16}-\frac{xy}{6}+\frac{y^2}{9}$ by $\frac{x}{4}-\frac{y}{3}$. . . $\frac{x}{4}-\frac{y}{3}$.

16. . . . . $\frac{x^3}{a^3}-\frac{z^3}{c^3}$ by $\frac{x}{a}-\frac{z}{c}$. . $\frac{x^2}{a^2}+\frac{xz}{ac}+\frac{z^2}{c^2}$.

PROPORTION.

Four quantities are proportional, when the first multiplied by any number contains the second, as often as the third multiplied by the same number contains the fourth; that is, if $\frac{ma}{b}=\frac{mc}{d}$, whatever number m represents, then the ratio of a to b is the same with that of c to d, or a is to b as c is to d; which is expressed thus, $a : b :: c : d$.

Hence three quantities may be proportional; for if $b=c$, then $a : b :: b : d$.

PROP. I.

The product of the extremes of four proportionals is equal to the product of *the means*, and conversely.

Let $a:b::c:d$, then (def. $m=1$) $\frac{a}{b}=\frac{c}{d}$; and multiplying both by bd, we obtain $\frac{abd}{b}=\frac{bcd}{d}$; and cancelling like quantities, $ad=bc$.

Conversely, if $ad=bc$, divide by bd, and $\frac{a}{b}=\frac{c}{d}$. Therefore, $a:b::c:d$.

If $ad=bc$, then $rsad=rsbc$; that is, $ra\times sd=rb\times sc$ $=sb\times rc$. Hence $ra:rb::sc:sd$, and $ra:sb::rc:sd$; so that if the terms of proportionals be multiplied or divided by any numbers, either the first and second by the same, or the first and third by the same number, the products or the quotients will be proportionals.

Hence, if three quantities be proportional, $a:b::b:d$, then $ad=b^2$, the product of the extremes is equal to the square of the mean. Hence, if $a:b::b:d$, then $a:d::aa:ad=bb$.

PROP. II.

Of four proportionals, any three being given, the fourth may be found.

Let a, b, c, be the three first, and x the fourth; then $ax=bc$, and dividing by a, we obtain $x=\frac{bc}{a}$; that is, the product of the means, divided by one of the extremes, gives the other extreme.

Hence, of three proportionals any two being given, the third may be found; for $ad=b^2$, therefore $b=\sqrt{ad}$, and $d=\frac{b^2}{a}$.

Let $7:10::28:x$. Required the value of x. Ans. 40.
Required a third proportional to 81 and 54. . 36.
Required a mean proportional between 49 and 4. 14.

PROP. III.

If $a:b::c:d$, and if $pa>qb$, then $pc>qd$; or if $pa=qb$, $pc=qd$; or if $pa<qb$, $pc<qd$.

Let $pa>qb$, take m, so that $mpa>(mq+1)b$, then mpc is not $<(mq+1)d$; that is, $mpc>mqd$, or $pc>qd$.

If $pa<qb$, the quotient of pa by b is less than that of qb by b; therefore pc divided by d is less than qd divided by d, or $pc<qd$.

If $pa=qb$, pc is not $>$ nor $<qd$; that is, $pc=qd$.

PROP. IV.

If $a:b::c:d$, then $a\pm b:b::c\pm d:d$; for $ma\div b=mc\div d$, and $mb\div b=md\div d$; therefore $ma\pm mb$, or $m(a\pm b)\div b=m(c\pm d)\div d$. Therefore $a\pm b:b::c\pm d:d$.

PROP. V.

If four quantities be proportionals, $a : b :: c : d$, they will be proportionals though they be altered in any of the following ways:—

1. $b : a :: d : c$, ... by Inversion;
2. $a : c :: b : d$, ... by Alternation;
3. $a+b : b :: c+d : d$, . . by Composition;
4. $a-b : b :: c-d : d$, . . by Division;
5. $a : a \pm b :: c : c \pm d$, . by Conversion;
6. $a+b : a-b :: c+d : c-d$, . by Mixing;

For in all these the product of the extremes may be shown to be equal to the product of the means.

PROP. VI.

If the terms of two proportions be multiplied or divided in their order, the products or quotients will be proportionals, as, $a : b :: c : d$, and $m : n :: p : r$; then $am : bn :: cp : dr$, and $\frac{a}{m} : \frac{b}{n} :: \frac{c}{p} : \frac{d}{r}$; for $ad = bc$ and $mr = np$, therefore $admr = bcnp$; that is, $am.dr = bn.cp$, &c.

Hence similar powers or roots of proportionals are proportionals.

OF NEGATIVE QUANTITIES.

If c be the difference between a and b, the algebraical expression for this is $a - b = c$, where a is supposed to be greater than b; if it be less, the expression is $a - b = -c$. As a greater quantity cannot be taken from a less, the expression $-c$ is impossible; so that a negative quantity standing by itself has, strictly speaking, no meaning. But if it be joined to another quantity, as $m - c$, the expression is proper, and may be subjected to all the operations of algebra. The absurdity appears only in the result; and when it does appear, it points out that something impossible has been admitted into the question, some condition inconsistent with its other conditions. We therefore reckon a negative result to be a proper algebraical solution of a problem, for it agrees with the preceding steps of the process, and points out the impossibility of the conditions, and thus it has its use in limiting the terms of the question. It will therefore be necessary in what follows to attend to negative expressions, and the forms which result from them, as well as from the positive ones. But this should create no hesitation in the operations; for it has been shown, not only how whole quantities, but also how single terms of them, may be added together or subtracted from one another,

and how they may be multiplied or divided by one another with the signs of the resulting terms. But it is to be remarked, that these signs do not belong to the terms taken as isolated quantities, but to the relation in which they stand to the other terms of the result. When Diophantus of old said, "A defect drawn into a defect produces an excess," he did not by *a defect* mean a simple quantity, without relation to any other quantity: he meant to express by it what one quantity wanted to make it equal to another, and that after the sum of the products of the wholes by these defects had been subtracted from the product of the wholes, the true product would exceed the remainder by the product of the defects, which must therefore be added to the remainder. And that this is the case, has been proved before, in the note explaining Multiplication. It is therefore improper to apply to simple quantities the rules by which the terms of compound quantities are connected together; and much of the obscurity of algebra has arisen from this confusion.

If $a-x$ be multiplied by itself, the product is $a^2-2ax+x^2$; and if $x-a$ be multiplied by itself, the product is the same; so that from this product it cannot be determined whether a be greater or less than x; that is, if $a-x=c$, whether the product has arisen from $+c$ or from $-c$, for each of these multiplied by itself produces $+c^2$, and therefore the square root of $+c^2$ may be either $+c$ or $-c$, and of course the square root of $-c^2$ is impossible. This expression is in some instances found useful for promoting the investigation of rules.

The formula $a^2-b^2=(a+b)\times(a-b)$ is useful in every branch of the mathematics. Now $a^2+b^2=a^2-b^2\times-1$ $=(a+b\sqrt{-1})\times(a-b\sqrt{-1})$. This latter expression is therefore useful in several investigations.

The algebraist does not consider the solution of a problem to be complete, unless it exhibit all the cases which can occur; and the results which flow from contradictory suppositions can only be exhibited by such expressions as have been just now explained.

In the application of algebra to various sciences, where position and other states must be introduced, quantities are often found in such opposite states, that when in one of them they are to be added, they must be invariably subtracted in the other. These different states may therefore be naturally pointed out by prefixing the sign $+$ to the quantity when it is in one of them, and the sign $-$ when it is in the opposite state; and this use does not appear to alter in the smallest degree the meaning affixed to these signs in the definitions, for

here they are prefixed solely for the purpose of subjecting the quantity to algebraical processes.

From the whole it appears, that the meaning of the signs + and — given in the definitions ought to be steadily adhered to, by which means many of the difficulties of beginners would be avoided.

In dividing a^5 by a^2, we either place the quantities in the form of a fraction, $\frac{a^5}{a^2}$, and expunge like quantities, which gives a^3 for the quotient, or else we subtract the exponent of the divisor from that of the dividend, $a^{5-2}=a^3$. These two methods make the quotients to have in some cases different appearances. Suppose a^2 to be divided by a^5. By the former method $\frac{a^2}{a^5}=\frac{1}{a^3}$. By the second $a^{2-5}=a^{-3}$; so that $a^{-3}=\frac{1}{a^3}$. Here the negative exponent does not represent a negative quantity, but only shows that the quantity placed in the numerator ought to be in the denominator; but in either place it can be subjected with equal ease to all the rules of algebra. From this it appears, that any quantity may be removed from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent. Thus $\frac{a^2b}{c^2}=a^2bc^{-2}$, $ab^{-3}c^2=\frac{ac^2}{b^3}$.

INVOLUTION.

Involution is the method of finding the powers of quantities.

RULE FOR SIMPLE QUANTITIES.

Multiply the exponent of each letter by the name of the power to which it is to be raised, and prefix the same power of the coefficient.

If the sign of the quantity be +, all its powers are positive; but if the sign be —, its odd powers have —, and all the rest have +.*

In a fraction, raise its terms separately to the power required.

1. Raise $+3ab^2$ to the 4th power. Ans. $+81a^4b^8$.
2. . . . $-2a^3x$ to the 6th power. . $+64a^{18}x^6$.

* It was shown in Multiplication, that $-x^m\times -x^m=+x^{2m}$, and $+x^{2m}\times -x^m=-x^{3m}$. Hence x^m raised to the nth power $=x^{mn}$, and $-x^m$ raised to the nth power is either $+x^{mn}$ or $-x^{mn}$, according as n is even or odd.

3. Raise $+\frac{4a^3bc^2}{3c}$ to the 5th power. Ans. $+\frac{1024a^{15}b^5c^{10}}{243c^5}$.

4. . . . $-\frac{7a^2}{3b^3}$ to the 3d power. . $-\frac{343a^6}{27b^9}$.

5. . . . $+\frac{2a^{\frac{1}{2}}b^{\frac{3}{4}}c}{3x^{\frac{1}{3}}v^{\frac{1}{2}}}$ to the 8th power. $+\frac{256a^4b^6c^8}{6561x^{\frac{8}{3}}v^4}$.

When the quantity is compound, raise it by actual multiplication.

Thus the powers of $a+b$ are,

2d, $=a^2+2ab+b^2$.

3d, $=a^3+3a^2b+3ab^2+b^3$.

4th, $=a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

5th, $=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

6th, $=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

The powers of $a-b$ are the same with those of $a+b$, except that the signs of the even terms are $-$, all the rest are $+$.

Hence it appears,

1. That the number of terms is one greater than the name of the power.

2. That the exponent of the leading quantity in the first term is the name of the power, and that it decreases by 1 in each of the following terms to the last, where it is 0.

3. That the second quantity is not found in the first term; in the second its exponent is 1; and it increases by 1 in each of the following terms to the last, in which it is the name of the power.

4. That the coefficient of the first term is 1, that of the second is the name of the power, and in the following terms it is got by multiplying the coefficient of the preceding term by the exponent of the leading quantity in that term, and dividing the product by the number of that term.

5. That when the signs of both quantities are alike, all the terms have the sign $+$; but if the signs of the quantities be different, the odd terms have $+$, and the even terms $-$.

1. Raise $x-v$ to the 7th power.

   Ans. $x^7-7x^6v+21x^5v^2-35x^4v^3+35x^3v^4-21x^2v^5+7xv^6-v^7$.

2. Raise $m-n$ to the 8th power.

   Ans. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.

3. Raise $ab - cd$ to the 5th power.

Ans. $a^5b^5 - 5a^4b^4cd + 10a^3b^3c^2d^2 - 10a^2b^2c^3d^3 + 5abc^4d^4 - c^5d^5$.

4. Raise $2a - 3b$ to the 4th power.

Ans. $(2a)^4 - 4(2a)^3(3b) + 6(2a)^2(3b)^2 - 4(2a)(3b)^3 + (3b)^4 = 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$.

NOTE.—In this manner care must be taken to distinguish the quantities affected by the different exponents, and to raise them accordingly.

5. Raise $8rs - 5vs$ to the 3d power.

Ans. $512r^3s^3 - 960r^2s^3v + 600rs^3v^2 - 125s^3v^3$.

6. Raise $x^2 - v^2$ to the 5th power.

Ans. $x^{10} - 5x^8v^2 + 10x^6v^4 - 10x^4v^6 + 5x^2v^8 - v^{10}$.

7. Raise $a^2 - 2ab$ to the 6th power.

Ans. $a^{12} - 12a^{11}b + 60a^{10}b^2 - 160a^9b^3 + 240a^8b^4 - 192a^7b^5 + 64a^6b^6$.

8. Raise $2ac - c^2$ to the 7th power.

9. . . . $3x^2 - 4xv$ to the 4th power.

10. . . . $5a^2c - 3xv^2$ to the 3d power.

11. . . . $a + b$ to the nth power.

Ans. $a^n + na^{n-1}b + n.\dfrac{n-1}{2}a^{n-2}b^2 + n.\dfrac{n-1}{2}.\dfrac{n-2}{3}a^{n-3}b^3$, &c. or dividing by a^n, and putting A, B, C, &c. for the preceding terms with their signs, it becomes $a^n \times \left(1 + \dfrac{nb}{a} + \dfrac{n-1}{2}.\dfrac{bA}{a} + \dfrac{n-2}{3}.\dfrac{bB}{a} + \dfrac{n-3}{4}.\dfrac{Cb}{a}, \text{ \&c.}\right)$ where the law of continuation is evident.

If the quantity consists of more than two terms, divide the terms into two classes, and raise them as if each class were a simple quantity; after which the classes must be raised according to the exponents placed over them, and then connected with one another, and with the coefficients by multiplication.

12. Raise $a + b - c$ to the 3d power.

Ans. $(a+b)^3 - 3 \times (a+b)^2c + 3(a+b)c^2 - c^3 = a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$.

13. Raise $a^2 + b^2 - c^2$ to the 2d power.

14. . . . $a^2 - 2ab + b^2$ to the 4th power.

15. . . . $a - b + c - d = (a - b) + (c - d)$ to the 3d power.

EVOLUTION.

Evolution is the method of finding the roots of quantities, or those from which given powers have been raised.

Rule.—In simple quantities, divide the exponents of the letters by the name of the root required, and prefix the same root of the coefficients.

If the sign of the given quantity be $+$, the sign of the root is also $+$. If the sign of the quantity be $-$, the sign of its odd roots is $-$; but it can have no even root, for the square of $+a$, and also of $-a$, is $+a^2$.*

1. Required the 3d root of a^6b^3. . Ans. a^2b.

2. . . . . 4th root of $\dfrac{16a^4b^6c^8}{81a^8}$. . $\dfrac{2ab^{\frac{3}{2}}c^2}{3a^2}$.

3. . . . . 5th root of $\dfrac{32a^{10}b^8c^5}{c^6x^3}$. . $\dfrac{2a^2b^{\frac{8}{5}}c}{c^{\frac{6}{5}}x^{\frac{3}{5}}}$.

4. . . . . 6th root of $\dfrac{m^3n^5}{c^6e^7}$. . . $\dfrac{m^{\frac{1}{2}}n^{\frac{5}{6}}}{ce^{\frac{7}{6}}}$.

TO FIND THE SQUARE ROOT OF A COMPOUND QUANTITY.

Take the square root of the first term for the first term of the root, and subtract its square from the given quantity. Double the root for a divisor, by which divide the next term to get another term of the root; annex this term with its proper sign to the divisor, and then multiply the divisor thus completed by it, and subtract the product from the resolvend, and proceed in the same way with the remainder.

1. Required the square root of $x^2-2xv+v^2$.

$$
\begin{array}{r|l}
 & x^2-2xv+v^2\ (x-v \text{ root} \\
 & x^2 \\
\hline
2x-v) & -2xv+v^2 \\
 & -2xv+v^2. \\
\end{array}
$$

---

* It was shown in the note on Involution, that x^{mn} is the nth power of x^m, therefore $x^{\frac{mn}{n}}$ is the nth root of x^{mn}, and consequently that $\frac{1}{n}$ is the proper exponent of the nth root; also that the nth power of $-x^m$ is either $+x^{mn}$ or $-x^{mn}$, according as n is even or odd. Therefore, in the first case, $+x^{\frac{mn}{n}}$, when n is even, may be either $+x^m$ or $-x^m$, and that in this case $-x^{\frac{mn}{n}}$ is impossible.

2. $\sqrt{x^4-2x^2+1}$ . . $=$ ns. x^2-1.

3. $\sqrt{\frac{x^2}{4}-xv+v^2}$ . . . $=$ $\frac{x}{2}-v$.

4. $\sqrt{x^4-4x^3a+6x^2a^2-4xa^3+a^4} = x^2-2xa+a^2$.

5. $\sqrt{\frac{a^2}{c^2}-\frac{2ax}{c}+x^2}$ . . $=\frac{a}{c}-x$.

6. $\sqrt{(a^2+2ab+b^2+2ac+2bc+c^2)} = a+b+c$.

7. $\sqrt{(4x^4+6x^3+\frac{89x^2}{4}+15x+25)} = 2x^2+\frac{3x}{2}+5$.

8. $\sqrt{(x^6+4x^5+2x^4+9x^2-4x+4)} = x^3+2x^2-x+2$.

TO EXTRACT ANY OTHER ROOT.

Arrange the terms as in Division; take the root of the first term for the first term of the root; raise this root to a power less by one than the given power, and multiply it by the name of the root for a divisor, by which divide the second term of the given quantity to get another term of the root. Raise the whole root thus found to the given power, and subtract it from the given quantity; and if there be a remainder, divide its first term by the divisor got before to obtain another term of the root, and proceed as before.

1. Required the cube root of $x^3+3x^2v+3xv^2+v^3$.

$$\begin{array}{rl} & x^3+3x^2v+3xv^2+v^3\,(\,x+v \\ & x^3 \\ 3x^2) & \overline{\quad +3x^2v} \\ (x+v)^3 = & \overline{x^3+3x^2v+3xv^2+v^3}. \end{array}$$

2. $(27a^3-54a^2c+36ac^2-8c^3)^{\frac{1}{3}} = 3a-2c$.

3. $(m^6+6m^5-40m^3+96m-64)^{\frac{1}{3}} = m^2+2m-4$.

4. $(16x^4-96x^3y+216x^2y^2-216xy^3+81y^4)^{\frac{1}{4}} = 2x-3y$.

5. $(81a^4-432a^3c+864a^2c^2-768ac^3+256c^4)^{\frac{1}{4}} = 3a-4c$.

6. $($ [illegible] $+\frac{10x^3v^2}{4}-\frac{10x^2v^3}{8}+\frac{5xv^4}{16}-\frac{v^5}{32})^{\frac{1}{5}} = x-\frac{v}{2}$.

[illegible] $(x$ [illegible] $-9x^5+\frac{135x^4}{4}-\frac{135x^3}{2}+\frac{1215x^2}{16}-\frac{729x}{16}+\frac{729}{64})^{\frac{1}{6}} =$ $x-1\frac{1}{2}$.

OF SURDS.

SURDS are expressions of the roots of such quantities as are not complete powers.

Thus $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is a surd, because a^2 is not a cube.

radical to the power or root required, and then to place the radical sign over it.

1. The 4th power of $\sqrt{3a}$ is $9a^2$.

2. The 3d power of $(a-b)^{\frac{1}{3}}$ is $a-b$.

3. The 4th power of $\frac{1}{6}\sqrt{6}$ is $\frac{1}{36}$.

4. The 5th power of $\frac{2\sqrt{a}}{3\sqrt[5]{c}}$ is $\frac{32\sqrt{a^5}}{243c}$.

5. The 3d root of $a^{\frac{1}{2}}b^{\frac{3}{4}}c$ is $a^{\frac{1}{6}}b^{\frac{1}{4}}c^{\frac{1}{3}}$.

6. The 4th root of $\frac{ab^{\frac{1}{2}}c^4}{a^3}$ is $\frac{a^{\frac{1}{4}}b^{\frac{1}{8}}c}{a^{\frac{3}{4}}}$.

7. The 3d root of $\frac{1}{8}\sqrt{2}$ is $\frac{1}{2}\sqrt[6]{2}$.

8. The 5th root of $\frac{b^{\frac{2}{3}}}{32a^{\frac{1}{3}}}$ is $\frac{1}{2}\sqrt[15]{\frac{b^2}{a}}$.

TO FIND THE SQUARE ROOT OF A COMPOUND SURD.

When a quantity consists of two terms, a rational and a surd; if it have a root, the rational part is the sum of the squares of its terms, and the surd is the double of their product.

Rule.—From the square of the rational term subtract the quantity affected by the radical sign, and take the square root of the remainder; add it to the rational term, and subtract it from that term, and take the halves of the sum and remainder for the squares of the two terms of the root.

1. $(6-\sqrt{20})^{\frac{1}{2}}=\sqrt{5}-1$, for $\sqrt{36-20}=\sqrt{16}=4$, and $\sqrt{\frac{6\pm4}{2}}=\sqrt{5}$ and 1.

2. $(136-96\sqrt{2})^{\frac{1}{2}}=6\sqrt{2}-8$.

3. $(51-10\sqrt{2})^{\frac{1}{2}}=5\sqrt{2}-1$.

4. $(14-6\sqrt{5})^{\frac{1}{2}}=3-\sqrt{5}$.

5. $(5-2\sqrt{6})^{\frac{1}{2}}=\sqrt{3}-\sqrt{2}$.

6. $(76-42\sqrt{3})^{\frac{1}{2}}=7-3\sqrt{3}$.

7. $(19+8\sqrt{3})^{\frac{1}{2}}=4+\sqrt{3}$.

8. $(12-2\sqrt{35})^{\frac{1}{2}}=\sqrt{7}-\sqrt{5}$.

9. $(7+4\sqrt{3})^{\frac{1}{2}} \quad = 2+\sqrt{3}.$

10. $(7-2\sqrt{10})^{\frac{1}{2}} \quad = \sqrt{5}-\sqrt{2}.$

11. $(39-6\sqrt{30})^{\frac{1}{2}} \quad = \sqrt{30}-3.$

EQUATIONS.

When two expressions are equal to one another, they are written with the sign $=$ of equality between them, and the whole is called an equation. Thus $x-a=b+c$ is an equation; $x-a$ is called the left side, and $b+c$ the right side of the equation.

REDUCTION.

Reduction is the method of bringing the unknown quantity to stand alone upon one side of the equation, and the known quantities upon the other. This is performed by the following rules taken in their order.

Rule 1.—If a term be divided by any quantity, multiply every term by the divisor.

In this way the equation may be cleared of fractions.

Rule 2.—Any term may be transposed from one side of the equation to the other, by changing its sign from $+$ to $-$, or from $-$ to $+$.

In this way the terms containing the unknown quantity may be brought to one side of the equation, and the known terms to the other; after which they may be collected into one by addition.

Cor.—If a term be found on both sides with the same sign, it may be erased from both.

Rule 3.—If the unknown quantity be multiplied by any other, divide both sides by the multiplier.

In this way the value of the unknown quantity is found, when there are no surds nor powers.

Rule 4.—If the equation have a surd in it, after bringing it to one side by itself, take away the radical sign, and raise the other side to the corresponding power.

Rule 5.—If one side of the equation be a complete power, take the corresponding root of both sides.*

* It is evident, that the operations prescribed in these rules do not render the two sides of the equation unequal, for they are both increased or diminished in the same degree. Thus, in the first operation, both sides are multiplied by the same quantity; in transposition the same quantity is subtracted from both sides; in the third both sides are divided by the same quantity; in the fourth they are both raised to the same power; and in the last the same root is taken of both sides.

1. Let the equation be $2x-\frac{19}{4}=\frac{3x}{4}+4$

Multiply by 4, . .	$8x-19=3x+16$
Add $19-3x$ to both sides,	$8x-3x=16+19$
And collecting, . . .	$5x=35$
Divide by 5, . . .	$x=7$

So that 7 is the value of x.

In the second line the equation is cleared of fractions, and in the third line the quantities 19 and $3x$ are transposed with their signs changed; and it is evident that the two sides of the equation have been kept equal to one another in every line.

2. Let the equation be $(3x+1)^{\frac{1}{2}}+5=10$

By transposing 5,	$(3x+1)^{\frac{1}{2}}=10-5=5$
Square by rule 4,	$3x+1=25$
Transposing 1, . .	$3x=25-1=24$
And dividing by 3, .	$x=8.$

The removal of the sign from the radical is equivalent to the raising of it to the power.

3. Let the equation be . $9x^2+9=3x^2+63$

By transposing, .	$9x^2-3x^2=63-9$
Collecting, . . .	$6x^2=54$
Dividing by 6, . .	$x^2=9$
Taking the square root, .	$x=3.$

REDUCE THE FOLLOWING EQUATIONS:

	EQUATIONS.	ANSWERS.
4.	$\frac{x}{2}-3=5.$	$x=16.$
5.	$6-x=4-\frac{2x}{3}.$	$x=6.$
6.	$4x-8=3x+20.$	$x=28.$
7.	$40-6x-16=120-14x.$	$x=12.$
8.	$x+\frac{x}{2}+\frac{x}{3}=11.$	$x=6.$
9.	$ax+2ab=3c^2.$	$x=\frac{3c^2}{a}-2b.$
10.	$5ax-3b=2dx+c.$	$x=\frac{3b+c}{5a-2d}.$
11.	$2x-\frac{x}{2}+1=5x-2.$	$x=\frac{6}{7}.$
12.	$x^{\frac{1}{2}}-2=6.$	$x=64.$

EQUATIONS.		ANSWERS.
13.	$(4x+16)^{\frac{1}{2}}=12$.	$x=32$.
14.	$5x-15=2x+6$.	$x=7$.
15.	$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=10$.	$x=9\frac{3}{13}$.
16.	$3x^2-x=8x+x^2$.	$x=4\frac{1}{2}$.
17.	$x-a=\frac{x^2}{x-a}$.	$x=\frac{a}{2}$.
18.	$(2x+3)^{\frac{1}{3}}+4=8$.	$x=30\frac{1}{2}$.
19.	$\left(\frac{2x}{3}\right)^{\frac{1}{2}}+5=7$.	$x=6$.
20.	$\frac{x-3}{2}+\frac{x}{5}=20-\frac{x-19}{2}$.	$x=25\frac{5}{6}$.
21.	$\frac{a}{1+x}+\frac{a}{1-x}=b$.	$x=\left(\frac{b-2a}{b}\right)^{\frac{1}{2}}$.
22.	$x+(a^2+x^2)^{\frac{1}{2}}=\frac{2a^2}{(a^2+x^2)^{\frac{1}{2}}}$.	$x=\frac{a}{\surd 3}$.
23.	$x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}=\frac{2a}{(a+x)^{\frac{1}{2}}}$.	$x=\frac{a}{3}$.
24.	$(12+x)^{\frac{1}{2}}=2+x^{\frac{1}{2}}$.	$x=4$.
25.	$(a^2+x^2)^{\frac{1}{2}}=(b^4+x^4)^{\frac{1}{4}}$.	$x=\left(\frac{b^4-a^4}{2a^2}\right)^{\frac{1}{2}}$.
26.	$bx^2+c+3=2bx^2+1$.	$x=\left(\frac{c+2}{b}\right)^{\frac{1}{2}}$.
27.	$4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$.	$x=11$.
28.	$a+x=[a^2+x(b^2+x^2)^{\frac{1}{2}}]^{\frac{1}{2}}$.	$x=\frac{b^2}{4a}-a$.
29.	$\frac{3x}{a}-c+\frac{x}{b}=4x+\frac{2x}{d}$.	$x=\frac{abcd}{(3b+a)d-2ab(2d+1)}$.
30.	$3x-\frac{a}{b}-cx=\frac{a+x}{3}-\frac{b-x}{a}$.	$x=\frac{a^2b-3b^2+3a^2}{8ab-3abc-3b}$.
31.	$5ax-2b+4bx=2x+5c$.	$x=\frac{5c+2b}{5a+4b-2}$.

EXTERMINATION OF UNKNOWN QUANTITIES.

When there are several unknown quantities, there must be as many equations: from these an equation must be deduced, which contains only one of the unknown quantities; and this equation is to be resolved by the preceding rules.

This elimination may be performed by any of the following methods:—

METHOD 1.—Find a value of one of the unknown quantities in each of the equations, supposing all the rest to be known. Make these values equal to one another, and from them find values of another unknown quantity. Make again these values equal, and find another unknown quantity, and so on, until an equation be obtained containing only one unknown quantity, which is to be resolved by the preceding rules.

METHOD 2.—Find a value of one of the unknown quantities in that equation in which it is least involved; substitute this value and its powers for that unknown quantity and its powers in all the other equations, and proceed in the same way with these equations to get rid of other unknown quantities.

METHOD 3.—Multiply the equations by such quantities as will make the coefficients of one of the unknown quantities, or of its highest power, the same in all the equations; then, if the signs of these equal terms be like, subtract the equations, but if the signs be unlike, add them, and new equations will arise, wanting that unknown quantity or its highest power, and these equations are to be treated in the same way.

NOTE.—The first method seems to be the most regular; the second is shorter than the first, but the reductions are more intricate; the third is the most simple and expeditious.

1. Let the equations be $x+y=12$, and $5x+3y=50$.

By Method 1. $x=12-y$, and $x=\frac{50-3y}{5}$; therefore $12-y=\frac{50-3y}{5}$: whence $y=5$, and $x=12-y=7$.

By Method 2. $5(12-y)+3y=50$ or $60-5y+3y=50$: whence $y=5$, and $x=7$.

By Method 3.
$$\begin{array}{r} 5x+5y=60 \\ \underline{5x+3y=50} \end{array}$$
By subtraction, $2y=10$

2. Exterminate $\left.\begin{array}{r} 5x+8y=124 \\ 3x-2y=20 \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=12 \\ y=8 \end{array}\right.$

3. . . . . . $\left.\begin{array}{r} 5x-3y=90 \\ 2x+5y=160 \end{array}\right\}$ . . $\left\{\begin{array}{l} x=30 \\ y=20 \end{array}\right.$

4. . . . . . $\left.\begin{array}{r} x-y=2 \\ 8y+5x-6y=120 \end{array}\right\}$ . . $\left\{\begin{array}{l} x=17\frac{5}{7} \\ y=15\frac{5}{7} \end{array}\right.$

5. . . . . . $\left.\begin{array}{r} \frac{x}{2}+\frac{y}{3}=16 \\ \frac{x}{5}-\frac{y}{9}=2 \end{array}\right\}$ . . $\left\{\begin{array}{l} x=20 \\ y=18 \end{array}\right.$

6. Exterminate $\left.\begin{array}{l} x + y = a \\ x^2 - y^2 = b \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = \frac{1}{2}a + \frac{b}{2a} \\ y = \frac{1}{2}a - \frac{b}{2a} \end{array}\right.$

7\. . . . . $\left.\begin{array}{l} 4x + 3y = 31 \\ 3x + 2y = 22 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 4 \\ y = 5 \end{array}\right.$

8\. . . . . $\left.\begin{array}{l} 5x - 4y = 19 \\ 4x + 2y = 36 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 7 \\ y = 4 \end{array}\right.$

9\. . . . . $\left.\begin{array}{l} 3x + 7y = 79 \\ 2y - \frac{x}{2} = 9 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 10 \\ y = 7 \end{array}\right.$

10\. . . . . $\left.\begin{array}{l} \frac{x+y}{3} + 1 = 6 \\ \frac{x-y}{7} + 3 = 4 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 11 \\ y = 4 \end{array}\right.$

11\. . . . . $\left.\begin{array}{l} \frac{x+y}{3} - 2y = 2 \\ \frac{2x-4y}{5} + y = \frac{23}{5} \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 11 \\ y = 1 \end{array}\right.$

12\. . . . . $\left.\begin{array}{l} \frac{3x-7y}{3} = \frac{2x+y+1}{5} \\ 8 - \frac{x-y}{5} = 6 \end{array}\right\}$. $\left\{\begin{array}{l} x = 13 \\ y = 3 \end{array}\right.$

13\. . . . . $\left.\begin{array}{l} x + y = 13 \\ x + z = 14 \\ y + z = 15 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = 6 \\ y = 7 \\ z = 8 \end{array}\right.$

14\. . . . . $\left.\begin{array}{l} 2x + 3y + 4z = 29 \\ 3x + 2y + 5z = 32 \\ 4x + 3y + 2z = 25 \end{array}\right\}$. $\left\{\begin{array}{l} x = 2 \\ y = 3 \\ z = 4 \end{array}\right.$

15\. . . . . $\left.\begin{array}{l} x + 100 = y + z \\ y + 100 = 2x + 2z \\ z + 100 = 3x + 3y \end{array}\right\}$. $\left\{\begin{array}{l} x = 9\frac{1}{11} \\ y = 45\frac{5}{11} \\ z = 63\frac{7}{11} \end{array}\right.$

16\. . . . . $\left.\begin{array}{l} x + y = 90 - z \\ 2x + 40 = 3y + 20 \\ x + 20 = 2z + 5 \end{array}\right\}$. $\left\{\begin{array}{l} x = 35 \\ y = 30 \\ z = 25 \end{array}\right.$

17\. . . . . $\left.\begin{array}{l} x + y + z = 53 \\ x + 2y + 3z = 105 \\ x + 3y + 4z = 134 \end{array}\right\}$. $\left\{\begin{array}{l} x = 24 \\ y = 6 \\ z = 23 \end{array}\right.$

18\. . . . . $\left.\begin{array}{l} \frac{1}{2}x + \frac{1}{3}v + \frac{1}{4}z = 62 \\ \frac{1}{3}x + \frac{1}{4}v + \frac{1}{5}z = 47 \\ \frac{1}{4}x + \frac{1}{5}v + \frac{1}{6}z = 38 \end{array}\right\}$. $\left\{\begin{array}{l} x = 24 \\ v = 60 \\ z = 120 \end{array}\right.$

QUADRATIC EQUATIONS.

If, after all the unknown quantities, except one, are exterminated from an equation, both that unknown quantity and its square are found in it, the equation is called a Quadratic.

TO RESOLVE A QUADRATIC EQUATION.

Having cleared the equation, and brought the terms involving the unknown quantity to one side of it by themselves, divide by the coefficient of the square of the unknown quantity, if it have one; then add to both sides the square of half the coefficient of the unknown quantity, which will complete the square of the side containing the unknown quantity; after which extract the square root of both sides, and the equation will be reduced to a simple one, which may be resolved as before.

Note 1.—Since the square root of $x^2-2ax+a^2$ is either $a-x$ or $x-a$, the root of the known side of the equation must have both the signs $+$ and $-$ before it. Sometimes both these give proper solutions, and at other times only one of them.

Note 2.—The root of the side involving the unknown quantity consists of that quantity, and of $\frac{1}{2}$ its coefficient with its sign.*

1. Let the equation be $3x^2+12x=96$
By dividing by 3, . $x^2+4x=32$
Add the square of 2, $x^2+4x+4=36$
And taking the root, . $x+2=\pm 6$
And transposing, . $x=\pm 6-2=+4$ or -8.

Here the positive value of the root only is proper.

2. Let the equation be $2x^2-8x=90$
Dividing by 2, . $x^2-4x=45$
Completing the square, $x^2-4x+4=49$
Taking the root, . . $x-2=\pm 7$
Transposing, . . . $x=\pm 7+2=+9$ or -5.

Here also the root 7 is greater than $\frac{1}{2}$ the coefficient of x; therefore the positive value only is proper.

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* Quadratic equations assume one of these three forms, viz. $x^2+ax=+b$; $x^2-ax=+b$; or $x^2-ax=-b$; and when they are resolved by the rule, the value of x assumes one of these forms, $x=\frac{-a\pm\sqrt{a^2+4b}}{2}$; $x=\frac{+a\pm\sqrt{a^2+4b}}{2}$; or $x=\frac{+a\pm\sqrt{a^2-4b}}{2}$.

If a positive answer is required, the sign of the radical in the first two forms must be $+$, but in the third it may be either $+$ or $-$. There is, however, a limitation in this case, for $4b$ must not be greater than a^2, otherwise the quantity below the radical sign would be negative, and its root impossible. This happens when the absolute term b is greater than $\frac{1}{4}a^2$, the square of $\frac{1}{2}$ the coefficient of x.

3. Let the equation be $15x - x^2 = 54$

or $x^2 - 15x = -54$

Completing the square, $x^2 - 15x + \frac{225}{4} = \frac{225}{4} - 54 = +\frac{9}{4}$

Taking the root, . . . $x - \frac{15}{2} = \pm\frac{3}{2}$

Transposing, ... $x = +\frac{15}{2} \pm \frac{3}{2} = +9$ or $+6$.

Here both the roots are proper. But it is to be remarked, that if 54 had been greater than $\frac{225}{4}$, the known side would have been negative, and its root impossible; in which case x would have had no value in numbers.

NOTE.—To avoid fractions, instead of dividing by the coefficient of x^2, and then adding the square of $\frac{1}{2}$ the coefficient; multiply the equation by 4 times the coefficient of x^2, and then add the square of the coefficient, which x had before multiplying.

4. Let the equation be $7x^2 - 20x = 32$

Multiplying by $4 \times 7 = 28$, $196x^2 - 560x = 896$

Adding $400 = 20^2$, . $196x^2 - 560x + 400 = 1296$

Taking the root, . . . $14x - 20 = \pm 36$

Whence . . . . . $x = +4$ or $-1\frac{1}{7}$.

EQUATIONS.	ANSWERS.
5. $8 + x^2 - 6x = 80$.	$x = +12$.
6. $8x - 20 = 70 - 2x^2$.	$x = 5$.
7. $3x^2 + 6 = 3x + 5\frac{1}{3}$.	$x = \frac{2}{3}$ or $\frac{1}{3}$.
8. $\frac{x}{3} + 42\frac{2}{3} = \frac{x^2}{2} + 20\frac{1}{2}$.	$x = 7$.
9. $3x^2 - 9 = 76 - 2x$.	$x = 5$.
10. $x^2 - x = 210$.	$x = 15$.
11. $\frac{1}{2}x^2 + 7\frac{3}{8} = \frac{1}{3}x + 8$.	$x = 1\frac{1}{2}$.
12. $4x^2 - 3x = 85$.	$x = 5$.
13. $\frac{4x^2}{3} - 11 = \frac{x}{3}$.	$x = 3$.
14. $5x^2 + 4x = 273$.	$x = 7$.
15. $\frac{7}{x+1} + \frac{2}{x} = 5$.	$x = \frac{2+\sqrt{14}}{5}$.
16. $\frac{x}{5} + \frac{5}{x} = 5\frac{1}{5}$.	$x = 25$ or 1.
17. $\frac{3x}{x+2} - \frac{x-1}{6} = x - 9$.	$x = 10$.

EQUATIONS.		ANSWERS.
18. $x^2+6ax=c^2$,	.	$x=(c^2+9a^2)^{\frac{1}{2}}-3a$.
19. $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$.	. .	$x=1\pm\sqrt{1-a^2}$.

NOTE.—If the equation contain two powers of the unknown quantity, and the exponent of the one is double that of the other, it may be resolved like a quadratic.

20. Let the equation be $x^6-6x^3=16$ Ans. $x=2$.
Completing the square, $x^6-6x^3+9=25$
Taking the root, . . $x^3-3=\pm 5$
Transposing, . . . $x^3=3\pm 5=8$
Taking the cube root, . . $x=2$.

EQUATIONS.		ANSWERS.
21. $2x^4-x^2=496$.	.	$x=4$.
22. $x^4+2ax^2=b$.	. .	$x=(\sqrt{a^2+b}-a)^{\frac{1}{2}}$.
23. $x-8x^{\frac{1}{2}}=9$.	. .	$x=81$ or 1.
24. $x-x^{\frac{1}{2}}=a$.	. .	$x=a+\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$.
25. $\frac{1}{2}x-\frac{1}{3}x^{\frac{1}{2}}=22\frac{1}{3}$.	.	$x=49$.
26. $(1+x)^{\frac{1}{2}}-2(1+x)^{\frac{1}{4}}=4$.		$x=55\pm 24\sqrt{5}$.
27. $3x^{2n}-2x^n=25$.	.	$x=\left(\frac{1\pm 2\sqrt{19}}{3}\right)^{\frac{1}{n}}$.
28. $x^n-6x^{\frac{n}{2}}=e$.	. .	$x=(18+e\pm 6\sqrt{e+9})^{\frac{1}{n}}$.
29. $4ax^4-bx^2=c$.	.	$x=\left(\frac{b\pm\sqrt{16ac+b^2}}{8a}\right)^{\frac{1}{2}}$.

SOLUTION OF QUESTIONS.

WHEN a question is proposed, the analyst ought to form a clear idea of its nature, and then attempt to express its terms, and the relations of its parts, in algebraical characters, putting the letters x, y, z, &c. for the unknown quantities in it; and in this way he must deduce as many independent equations from the conditions of the question as there are unknown quantities in it, which he can always do when the question is properly limited; after which, these equations being resolved by the preceding rules, will give the answer or answers.

Suppose x the greatest unknown quantity, y the next, z, v, &c. the lesser ones in their order.

Suppose it to be a condition of the question, that

The two quantities together, or their sum, amounts to 18. This condition may be expressed thus, . $x+y=18$

Their excess, difference, &c. is 6, . $x-y=6$

Their product, rectangle, the one into the other, or multiplied by it, is 72, . . $xy=72$

One of them taken out of the other, divided by it, applied to it, or their quotient, is 2, $\frac{x}{y}=2$

The greater is to the less, or their ratio is as 4 to 2, ... $x:y::4:2$

This proportion, by multiplying the means and the extremes, becomes an equation, . $2x=4y$

The sum of their squares is 180, . $x^2+y^2=180$

The difference of their squares is 108, $x^2-y^2=108$

And in a similar way may any other relations of the quantities be expressed in equations.

When the relation of one unknown quantity to another is simple, a letter may be taken for one of them, and an expression for the other deduced from the relation between them, which will abridge the work, and render it more elegant. Thus, if their difference be 3, take y for the less, and $y+3$ will be the greater.

It will often abridge the work, if letters are taken not for the unknown quantities themselves, but for their sum, difference, or any other relation from which the quantities may be easily found.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find such a number, that, if it be multiplied by 5, and also by 3, the former product shall exceed the latter by 26. The first product is $5x$, the second $3x$, their difference $5x-3x=26$. Ans. 13.

2. To find a number, to which if 27 be added, the sum shall be 10 times the number required.

$$10x=x+27. \qquad \text{Ans. 3.}$$

3. To find a number, from which if 4 be taken, and the remainder multiplied by 3, the product shall be twice the number sought. $(x-4)\times 3=2x$. Ans. 12.

4. To find a number of which the fourth part exceeds the fifth part by 13.

$$\frac{x}{4}-\frac{x}{5}=13. \qquad \text{Ans. 260.}$$

5. To find a number, to the half of which if 7 be added,

the sum shall be equal to twice the number with 20 taken from it.* Ans. 18.

6. To find a number of which the square shall be equal to 4 times the number, together with 5 times the same number. Ans. 9.

7. To find a number, to which if its half, its third, and its fourth parts be added, the sum shall be equal to the square of that number.

$$x^2 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4}.$$ Ans. $2\frac{1}{12}$.

8. To find a number, from which if 3 be taken, and the remainder multiplied by 3, and then 4 added to the product, the sum divided by 5 shall give half the number sought. Ans. 10.

9. To find a number of pounds, to which if 3 be added, and the sum multiplied by 12, the product shall be equal to the number of shillings in the value of the pounds, diminished by as many crowns as there are pounds required. Ans. £12.

10. To find two numbers, of which the sum is 133, and their difference 47.

y, and $y+47$, the numbers, $2y+47=133$. Ans. 90 and 43.

11. To find two numbers of which the sum is 84, and their quotient 13. Ans. 78 and 6.

12. To find two numbers of which the difference is 104, and their quotient 9. Ans. 117 and 13.

13. To find two numbers, so that 3 times the greater added to twice the less shall make 54, and 4 times the greater with 3 times the less shall make 75. Ans. 12 and 9.

14. To find two numbers, so that the greater with half the less shall make 25, and the less with half the greater shall make 23. Ans. 18 and 14.

15. To find two numbers in the ratio of 4 to 3, so that if one be added to each of them, the sums shall be in the ratio of 9 to 7.

$$3x = 4y,\ (x+1)\times 7 = (y+1)\times 9.$$ Ans. 8 and 6.

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* The intention of this section is to assist the learner in transferring the conditions of the question from common language into algebraic expressions, and thus forming equations, which are to be solved by the three preceding sections. The equations were inserted in the first edition, as being the proper answers aimed at; but many eminent teachers have suggested that this has a tendency to prevent students from exerting their own powers. They are now therefore omitted, except where some difficulty is apprehended in forming them, and which might not easily be got over without assistance.

16. To find two numbers of which the difference shall be 9, and the difference of their squares 351. Ans. 24 and 15.

17. To divide the number 36 into two parts, so that the square of the greater part shall exceed that of the less by 360. Ans. 23 and 13.

18. To divide the number 72 into two parts, so that three times the greater shall exceed twice the less by 121. Ans. 53 and 19.

19. To divide the number 56 into two parts, which shall be to one another as 4 to 3. Ans. 32 and 24.

20. To find a number, so that its half added to its third part shall be greater by $6\frac{1}{2}$ than its double divided by 5. Ans. 15.

21. To find a number, from the double of which if 22 be taken, the remainder shall exceed 100 as much as the number itself is below 100.

$$2x - 22 - 100 = 100 - x.$$ Ans. 74.

22. A person being asked his age, replied, that $\frac{1}{4}$ of his age, multiplied by $\frac{1}{6}$ of his age, would produce his age. How old was he? Ans. 30.

23. A general sends out $\frac{1}{3}$ of his army, and 1500 men more, and he retains $\frac{1}{2}$ of his army, and 1200 men more. How many men had he in his army? Ans. 16,200.

24. A gentleman distributing money among some poor people, found that he wanted 10s. to be able to give 5s. to each of them; he therefore gave each 4s., and then he had 5s. left. How much money had he, and how many poor were there? Ans. 15 poor, 65s.

25. To find two numbers in the ratio of 3 to 2, so that their sum shall be the sixth part of their product. Ans. 15 and 10.

26. There were 6 children in a family, whose ages differed by 2 years, and each received a guinea for every year of his age, and the money they received amounted to 72 guineas. Required their ages? Ans. 7 youngest, 17 eldest.

27. A and B inherited equal estates; but A spent annually £60 more than his income, while B saved £80 annually; in consequence of which, at the end of 12 years, B was twice as rich as A. Required the value of their estates? Ans. £2400.

28. A says to B, If you will give me £25, I shall have as much money as you shall have left. Says B, If you give me £30, I shall then have twice as much as you will have remaining. How much had each? Ans. B £190, A £140.

29. A farmer has 15 more cows than horses, and as many

scores of sheep as horses and cows together; the number of all the three is 651. How many has he of each kind?

Ans. 8 horses, 23 cows, 31 scores sheep.

30. Two merchants join in company with a capital of £2000. A's share was 11 months in trade, and B's 9 months, and their shares of the gain were equal. What was the stock of each? Ans. B's £1100, A's £900.

31. A field was sown with wheat at 35s. per boll, and produced 9 returns: the crop was sold at 30s. per boll, and, after paying for the seed, there remained £293, 15s. How much wheat was sown? Ans. 25 bolls.

32. A merchant laid aside £200 annually for his expenses, and increased his capital annually by $\frac{1}{3}$ of what was not thus expended. At the end of three years his capital was double of what he began with. What was it at first?

$$x+\frac{x-200}{3}+\frac{4x-800}{9}+\frac{16x-3200}{27}=2x. \qquad \text{Ans. £740.}$$

33. Five persons have money divided among them. The share of the first was £10 more than that of the second; the share of the second was £16 less than that of the third; the share of the third was £5 more than that of the fourth; and the share of the fourth £15 less than that of the fifth: also the shares of the two last were together equal to the sum of the shares of the other three. What was the share of each?

Ans. £21, £11, £27, £22, £37.

34. Two travellers set out at the same time to meet one another, from two places distant 390 miles: the first travels 30 miles in a day, and the other 22 miles. In what time will they meet? Ans. $7\frac{1}{2}$ days, 225 miles, 165 miles.

35. A privateer, sailing at the rate of 9 miles in an hour, discovers a merchant vessel 18 miles distant, sailing at the rate of 7 miles in an hour. In what time will the privateer overtake the other vessel? Ans. 9 hours.

36. A woman bought some apples at 3 for a penny, and as many at 2 for a penny, and sold them all again at 5 for twopence, and found that she had lost sixpence. How many of each kind did she buy? Ans. 180.

37. A hare, 40 of her leaps before a hound, takes 4 leaps for the hound's 3, but 2 of the hound's leaps are equal to 3 of the hare's. How many leaps must the hound take before he catch the hare?

$$\frac{3x}{2}-\frac{4x}{3}=40. \qquad \text{Ans. 240 hound's leaps.}$$

38. A son asked his father's age. The father replied, 7 years ago I was 3 times as old as you were; but if we live

together 7 years longer, my age will be the double of yours. What were their ages? Ans. 49 and 21.

39. An army being drawn up in a square, there were 79 men over; but in attempting to enlarge each side of the square by one man, there were 80 men too few. Required the number of men? Ans. 6320 men.

40. The paving of a square court, at 8d. per square yard, cost as much as the enclosing of it at 5s. the yard. Required its extent? Ans. 30 yards each side.

41. A person lost $\frac{1}{5}$ of his money by gaming, and then won 4s. Again he lost $\frac{1}{4}$ of what he then had, and afterwards won 3s. The third time he lost $\frac{1}{3}$ of what he then had; and after that, he had remaining $\frac{1}{2}$ of what he began with. How much money had he?

$$\frac{4x}{5}+4-\frac{4x}{20}-1+3-\frac{2x}{10}-2=\frac{x}{2}. \qquad \text{Ans. 40s.}$$

42. A cistern can be filled with water by one cock in 12 hours, and by another in 8 hours. In what time will it be filled if both run together? Ans. $4\frac{4}{5}$ hours.

43. The tail of a fish weighed 9 lb., the head weighed as much as the tail and half the body, and the weight of the body was equal to that of the head and tail. What was the weight of the fish? Ans. 72 lb.

44. A gentleman's two horses with the harness cost him £120; the value of the worst horse with the harness was double that of the best horse, and the value of the best horse with the harness was triple that of the worst horse. What was the value of each?

Ans. £50 harness, £40 and £30 horses.

45. A master with his apprentice can perform a piece of work in 8 days, which the master alone could do in 12 days. In what time could the apprentice do it?

$$\frac{x}{8}-\frac{x}{12}=1. \qquad \text{Ans. 24 days.}$$

46. Three men can do a piece of work, the first in 50 hours, the second in 60 hours, and the third in 75 hours. In what time will they do it, all working together? Ans. 20 hours.

47. A and B together can do a piece of work in 12 hours, A and C together in 20 hours, and B and C together in 15 hours. In what time will they do it, all working together, and in what time will each do it separately? $x=$ time all take.

$$\frac{x}{12}+\frac{x}{20}+\frac{x}{15}=2. \qquad \text{Ans. Together 10 hours, A 30, B 20, C 60.}$$

48. A labourer engages to work 160 days, on condition that he should receive half-a-crown for every day that he wrought, and should forfeit 10d. for every day he was absent from work. At the end of the stipulated time he had nothing to receive nor to pay. How many days did he work?

Ans. Wrought 40 days.

49. To find three numbers, so that the first with $\frac{1}{2}$ of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, shall each be equal to 34.

Ans. 10, 22, and 26.

50. To find a number consisting of three places, of which the digits have equal differences in their order, and if the number be divided by the sum of its digits, the quotient shall be 48; and if 198 be subtracted from the number, the digits shall be inverted. $100x+10y+z$ the number.

$x+z=2y$, $48\times 3y=$ number, $99x-99z=198$. Ans. 432.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

51. To divide the number 100 into two parts, so that their product shall be 2100. Ans. 70 and 30.

52. To find two numbers of which the difference shall be 8, and their product 240. Ans. 20 and 12.

53. To find two numbers of which the difference shall be 12, and the sum of their squares 1424. Ans. 32 and 20.

54. To find two numbers of which the sum shall be 30, and their product 224. Ans. 16 and 14.

55. To find two numbers of which the product shall be 108, and the sum of their squares 225. Ans. 12 and 9.

56. A gardener and his lad digged each a square piece of ground, of which the side was as many feet long as the worker was years old. The difference of their ages was 12 years, and the number of square feet digged by both was 1040. Required their ages? Ans. 28 and 16.

57. An oblong pond was surrounded by a terrace-walk 7 yards broad, the pond measured 15000 square yards, and the walk 3696 square yards. Required the length and breadth of the pond?

$$xy=15000, \text{ and } 14x+14y+196=3696.$$

Ans. 150 and 100 yards.

58. To find two numbers of which the sum is 13, and the sum of their cubes 637. Ans. 8 and 5.

59. To find two numbers of which the product shall be 120, and the product of the greater, increased by 8, multiplied by the less, increased by 5, shall be 300.

Ans. 12 and 10, or 16 and $7\frac{1}{2}$.

60. To divide 125 into two parts, so that the sum of their square roots shall be 15.

$$\sqrt{y}+(125-y)^{\frac{1}{2}}=15.$$ Ans. 100 and 25.

61. A grazier bought a number of sheep for £60, and, reserving 15 to himself, he sold the remainder for £54, and gained 2s. on each of them. How many sheep did he buy, and what did each cost? Ans. 75 sheep at 16s.

62. Sold an ox for £24, and gained as much per cent. as the ox cost. What was paid for him?

$$x+\frac{x^2}{100}=24.$$ Ans. £20.

63. A person bought some oxen for £80: if he had got 4 oxen more for the same money, each of them would have cost him £1 less. How many did he buy? Ans. 16.

64. A number of bees alighted upon a tree: at the first flight the square root of $\frac{1}{2}$ of them went away, and at the next $\frac{8}{9}$ of them, and then only two bees remained. How many alighted on the tree?

$$\sqrt{\tfrac{1}{2}x}+\frac{8x}{9}+2=x.$$ Ans. 72 bees.

65. A person bought cloth for £33, 15s., which he sold again at £2, 8s. per piece, and gained as much as a piece cost him. Required the number of pieces? Ans. 15 pieces.

66. A and B set out at the same time for a place at the distance of 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end $8\frac{1}{3}$ hours before him. At what rate per hour did each person travel?

Ans. A 9 miles, B 6 miles.

67. There are two numbers, of which the product is 120: if 2 be added to the less, and 3 subtracted from the greater, the product of the sum and difference will be also 120. What are the numbers? Ans. 15 and 8.

68. A and B distribute each £1200 among some poor persons: A relieves 40 persons more than B, and B gives £5 a-piece to each person more than A. How many persons were relieved by A and B? Ans. 120 by A, 80 by B.

69. A person bought some sheep for £57, but he lost 8 of them, and then sold the remainder at 8s. a-head profit; and thus he neither gained nor lost by the bargain. How many sheep did he buy? Ans. 38.

70. To divide the number 18 into two factors, so that the sum of their cubes shall be 243. Ans. 6 and 3.

71. There is a number consisting of two digits, the left-hand digit is 3 times the other; and if 12 be subtracted from the

number, the remainder will be the square of the left-hand digit. What is the number? Ans. 93.

72. A, travelling to London, overtook at the 50th milestone a flock of sheep, proceeding at the rate of 3 miles in 2 hours; and 2 hours afterwards met a waggon moving at the rate of 9 miles in 4 hours. B, travelling at the same rate, overtook the sheep at the 45th milestone, and met the waggon 40 minutes before he came to the 31st milestone. Where would B be when A reached London? $x =$ distance between them, $y =$ rate of their travelling per hour, $\frac{10y}{3} - 5 = x$, $50 - 2y - \frac{32y^2}{27} + \frac{76y}{9} = 31 + \frac{2y}{3} - x$. Ans. $x = 25$, $y = 9$.

LITERAL ANALYSIS.

When the known quantities are expressed in numbers, these numbers disappear during the progress of the operation, and the answer, when obtained, does not exhibit the process by which it has been deduced from the assumed data. This is the mode of solution given in the preceding parts of this work, and it was necessary for beginners; but it does not exhibit sufficiently the true difference between arithmetic and algebra, but rather confounds them. The essential character of algebra, taken in its most extensive meaning, is, that the results of its operations do not give the particular values of the quantity or quantities sought; they only represent the operations which ought to be made upon the given quantities, for obtaining the values of those sought, according to the conditions of the problem; so that the principal object of algebra is the investigation of theorems and the exhibition of rules for the arithmetical or geometrical solution of problems. For accomplishing these purposes, it is necessary to represent the known quantities by letters, as well as the unknown ones. The former are represented by the first letters of the alphabet, a, b, c, &c. and the unknown ones by the last letters, x, y, z, &c. The question is translated into equations, and these equations are resolved by the preceding rules; and then the values of the unknown quantities will be expressed in a general way, from their relations to those which are given in the question. Consequently, if this general expression be transferred from algebraical characters into common language, it will give a general rule for the solution of all questions of the same kind. But the expressions will answer the same purpose as accurately in algebraical characters, and then they are called Theorems, or Formulæ.

1. Given the sum s, and the difference d, of two quantities

x and y: to find the quantities. $x+y=s$, and $x-y=d$: by adding these equations we get $2x=s+d$, whence $x=\frac{s+d}{2}$; and by subtracting the equations we get $2y=s-d$, and $y=\frac{s-d}{2}$. These values, expressed in common language, give the following rules, viz.

To find the greater, add the difference to the sum, and divide by 2.

To find the less, subtract the difference from the sum, and divide by 2.

2. Given the sum s, of two quantities x and y, and the difference of their squares $=$ D, to find the quantities. $x+y=s$, and $x^2-y^2=\mathrm{D}$; and dividing the latter by the former, we get $x-y=\frac{\mathrm{D}}{s}$; whence, as before, $x=\frac{s}{2}+\frac{\mathrm{D}}{2s}$ and $y=\frac{s}{2}-\frac{\mathrm{D}}{2s}$, or $x=\frac{s^2+\mathrm{D}}{2s}$ and $y=\frac{s^2-\mathrm{D}}{2s}$.

3. As exercises, the student may investigate the following, viz. Of two quantities, their sum, difference, product, quotient, sum and difference of their squares, any two being given, to find all the rest. The operations will be similar to those used in the two last questions; and the results, except for the sum and difference of their squares, are given in the following Table, in which x and y are the quantities, $s=$ their sum, $d=$ their difference, $p=$ their product, $q=$ their quotient, $\mathrm{Z}=$ the sum of their squares, and $\mathrm{D}=$ the difference of their squares.

The use of this table is plain. Suppose the sum of two numbers to be 277, and their difference to be 115; then the greater number is $\left(\frac{s+d}{2}\right)=\left(\frac{277+115}{2}\right)=\frac{392}{2}=196$.

Suppose again the difference of two numbers to be 10, and their product 119.

The greater number is $\frac{d+(d^2+4p)^{\frac{1}{2}}}{2}=\frac{10+(100+476)^{\frac{1}{2}}}{2}=\frac{10+\sqrt{576}}{2}=\frac{10+24}{2}=17$.

Suppose the sum of their squares to be 250, and the difference of their squares to be 88.

The greater number is $\left(\frac{\mathrm{Z}+\mathrm{D}}{2}\right)^{\frac{1}{2}}=\left(\frac{250+88}{2}\right)^{\frac{1}{2}}=\sqrt{169}=13$.

The less is $\left(\frac{\mathrm{Z}-\mathrm{D}}{2}\right)^{\frac{1}{2}}=\left(\frac{250-88}{2}\right)^{\frac{1}{2}}=\sqrt{81}=9$.

TABLE.

Given.	Greater $= x$.	Less $= y$.	Sum $= s$.	Difference $= d$.	Product $= p$.	Quotient $= q$.
s and d	$\frac{s+d}{2}$	$\frac{s-d}{2}$			$\frac{s^2-d^2}{2}$	$\frac{s+d}{s-d}$
s and p	$\frac{s+(s^2-4p)^{\frac{1}{2}}}{2}$	$\frac{s-(s^2-4p)^{\frac{1}{2}}}{2}$		$(s^2-4p)^{\frac{1}{2}}$		$\frac{s+(s^2-4p)^{\frac{1}{2}}}{s-(s^2-4p)^{\frac{1}{2}}}$
s and q	$\frac{sq}{q+1}$	$\frac{s}{q+1}$		$\frac{q-1}{q+1}s$	$\frac{s^2q}{(q+1)^2}$	
d and p	$\frac{d+(d^2+4p)^{\frac{1}{2}}}{2}$	$\frac{d-(d^2+4p)^{\frac{1}{2}}}{2}$	$(d^2+4p)^{\frac{1}{2}}$			$\frac{d+(d^2+4p)^{\frac{1}{2}}}{d-(d^2+4p)^{\frac{1}{2}}}$
d and q	$\frac{dq}{q-1}$	$\frac{d}{q-1}$	$\frac{q+1}{q-1}\times d$		$\frac{qd^2}{(q-1)^2}$	
p and q	$(pq)^{\frac{1}{2}}$	$\sqrt{\frac{p}{q}}$	$(q+1)\sqrt{\frac{p}{q}}$	$(q-1)\sqrt{\frac{p}{q}}$		
d and D	$\frac{d^2+D}{2d}$	$\frac{D-d^2}{2d}$	$\frac{D}{d}$		$\frac{D^2-d^4}{4d^2}$	$\frac{D+d^2}{D-d^2}$
Z and D	$\left(\frac{Z+D}{2}\right)^{\frac{1}{2}}$	$\left(\frac{Z-D}{2}\right)^{\frac{1}{2}}$	$(Z+\sqrt{Z^2-D^2})^{\frac{1}{2}}$	$(Z-\sqrt{Z^2-D^2})^{\frac{1}{2}}$	$\frac{\sqrt{Z^2-D^2}}{2}$	$\frac{\sqrt{Z^2-D^2}}{Z-D}$

4. Given the sum s, of the products of two quantities, by known multipliers m and n, and also the sum of their products c, by other known multipliers p and q, to find the quantities.

Here $mx+ny=s$, and $px+qy=c$; and multiplying the former equation by p, and the latter by m, they become $pmx+pny=ps$, and $mpx+mqy=mc$; and subtracting, we get $npy-mqy=ps-mc$; and dividing by $np-mq$, we get $y=\frac{ps-mc}{np-mq}$; and in the same way we find $x=\frac{qs-nc}{np-mq}$.

5. Given the sum s, of the quotients of two quantities by known divisors m and n, and also the sum c, of their quotients by other known divisors p and q, to find the quantities.

Here $\frac{x}{m}+\frac{y}{n}=s$, and $\frac{x}{p}+\frac{y}{q}=c$, whence $nx+my=mns$, and $qx+py=pqc$; which, resolved as the last, gives $x=\frac{pmns-mpqc}{pn-qm}$, and $y=\frac{qmns-pqnc}{pn-qm}$.

6. Given the values m and n, of two ingredients, to find the quantities which must be taken of each, to form a given quantity a, of a compound of a given value e.

$$x+y=a, \text{ and } mx+ny=ae.$$

Ans. $x=\frac{e-n}{m-n}a$, and $y=\frac{e-m}{m-n}a$.

7. Given the times m and n, in which two agents could produce the same effect separately, to find the time in which they could do it jointly.

$$\frac{x}{m}+\frac{x}{n}=1. \qquad \text{Ans. } x=\frac{mn}{m+n}.$$

8. Given the times m, n, and r, in which three agents can perform the same work separately; to find the time in which they can do it jointly.

$$\frac{x}{m}+\frac{x}{n}+\frac{x}{r}=1. \qquad \text{Ans. } x=\frac{mnr}{mn+mr+nr}.$$

9. Given the times m, n, and r, in which every two of three agents can perform the same work; to find the time x, in which they can do it jointly, and also the times y, z, and v, in which each of them can do it separately.

Ans. $x=\frac{2mnr}{mn+mr+nr}$, $y=\frac{2mnr}{(m+n)r-mn}$, $z=\frac{2mnr}{(m+r)n-mr}$, and $v=\frac{2mnr}{(n+r)m-nr}$.

10. Given the specific gravities m and n, of two ingredients,

and the quantity a, of the mixture, with its specific gravity r; to find the quantities of the ingredients.

Ans. $x=\frac{ma}{r}\times\frac{r-n}{m-n}$, and $y=\frac{na}{r}\times\frac{m-r}{m-n}$.

11. Given the first distance d, of two moving bodies, and their velocities m and n; to find the time of their conjunction.

Ans. $x=\frac{d}{m-n}$.

12. Given the sum $2s$, of two numbers, and also the sum of their squares, of their cubes, of their fourth, or of their fifth powers, &c.; to find the numbers.

NOTE.—If their difference be $2x$, the numbers will be $s+x$ and $s-x$; and then the sum of their squares will be $2s^2+2x^2$, the sum of their cubes $2s^3+6sx^2$, the sum of their fourth powers $2s^4+12s^2x^2+2x^4$, and the sum of their fifth powers $2s^5+20s^3x^2+10sx^4$, all of which are of the quadratic or simple form, and may be resolved as before; but the sums of the higher powers exceed the quadratic.

Let $z=$ sum of their squares, $c=$ sum of their cubes, $q=$ sum of their fourth powers, and $p=$ sum of their fifth powers; then $x=\left(\frac{z-2s^2}{2}\right)^{\frac{1}{2}}=\left(\frac{c-2s^3}{6s}\right)^{\frac{1}{2}}=\left(-3s^2\pm\sqrt{\tfrac{1}{2}q+8s^4}\right)^{\frac{1}{2}}$
$=\left(-s^2\pm\sqrt{\frac{p}{10s}+\frac{4}{5}s^4}\right)^{\frac{1}{2}}$.

13. To find two numbers of which the product is given p, and also the product P, of the sums when each is increased by a given number (a and b).

Ans. $x=\frac{P-p-ab}{2b}\pm\sqrt{\left(\left(\frac{P-p-ab}{2b}\right)^2-\frac{ap}{b}\right)}$.

14. To find two numbers such, that their sum, their product, and the difference of their squares, shall be all equal.

Ans. $x=\frac{3+\sqrt{5}}{2}$.

15. Given the sum a, of two numbers, and the sum of their square roots b; to find the numbers.

Ans. $x=\frac{b\pm\sqrt{2a-b^2}}{2}$.

16. Given the excess of the product of two numbers above their sum a, and also the sum of their squares b; to find the numbers.

Ans. Let $m=\sqrt{(2a+b+1)}$;

then $x=\frac{1+m\pm\sqrt{b-2a-2-2m}}{2}$.

17. Given the sum s, of three numbers, of which the square of the greatest is equal to the squares of the other two, and also the continued product p, of the three numbers; to find the numbers.

Ans. The greatest is $\frac{s^2 \pm \sqrt{s^4 - 16sp}}{4s}$; the sum of the two lesser is $\frac{3s^2 \pm \sqrt{s^4 - 16sp}}{4s}$; and their product is $\frac{s^2 \pm \sqrt{s^4 - 16sp}}{4}$.

18. Let p, be the given product of the two lesser numbers, the rest as before; to find the numbers.

Ans. The greatest is $\frac{s^2 - 2p}{2s}$, and the sum of the two lesser ones is $\frac{s^2 + 2p}{2s}$.

19. Let, as before, the square of the greatest be equal to the squares of the other two, and the square of the middle one equal to the product of the greatest and least, and let the sum s, of the three be given; to find each of them.

Ans. The least $= \frac{2s}{3 \pm \sqrt{5} + 2\sqrt{\frac{1}{2} \pm \frac{1}{2}\sqrt{5}}}$.

20. Suppose still the square of the greatest equal to the squares of the other two, and let the difference of the squares of the two least be equal to the product of the greatest by a given multiplier m, also the difference of the two least is given $= d$; to find the numbers.

Ans. The greatest is $= \frac{d^2}{\sqrt{2d^2 - m^2}}$.

PROGRESSIONS.

A SERIES of quantities, which increase or decrease by a common difference, is called an Arithmetical Progression; as, 2, 5, 8, 11, &c., or 88, 85, 82, &c.

A series of quantities, which increase by a constant multiplier, or decrease by a common divisor, is called a Geometrical Progression; as, 2, 8, 32, 128, &c., or 567, 189, 63, &c.

The greatest and least terms are called the Extremes, and the other terms the Means.

ARITHMETICAL PROGRESSION.

If a represent the least term, y the greatest, d the common difference, and n the number of terms, any arithmetical progression may be expressed thus: a, $a+d$, $a+2d$, $a+3d$, &c. ascending; or y, $y-d$, $y-2d$, $y-3d$, &c. descending.

From these expressions it appears that the coeffic in any term is less by 1 than the number of that term

PROP. I.—The difference between the extremes is e the common difference, multiplied by the number o wanting one. For the coefficient of d in the nth term is

Cor.—Hence $y = a + (n - 1)d$, and $a = y - (n -$

PROP. II.—The sum of the extremes is equal to the s any two terms equally distant from them.

For any term exceeds the least, as much as its correspon term is less than the greatest. Thus, if half the series as from a, while the other half descends from y, the whole be a, $a + d$, $a + 2d$, &c., $y - 2d$, $y - d$, y; where the s of any two corresponding terms is $a + y$.

Cor.—The double of any term is equal to the sum of any two terms equally distant from it.

PROP. III.—The sum of any number of terms in arithmetical progression is equal to the sum of the extremes multiplied by half the number of terms.

For by adding the extremes, and every two equally distant from them, we obtain equal sums, of which the number is half the number of terms of the series.

Cor. 1.—Hence if $s =$ sum of the series, $s = (a + y)\frac{n}{2}$.

Cor. 2.—If the number of terms be odd, and m the middle one, then $s = nm$; for $2m = a + y$.

Cor. 3.—In a series of natural numbers, 1, 2, 3, &c. n, the sum $s = n \times \frac{n+1}{2}$; for n is the greatest term, and $n + 1$ the sum of the extremes.

Cor. 4.—In a series of even numbers, 2, 4, 6, &c., $s = n(n + 1)$; for this series is $2 \times (1 + 2 + 3)$, &c.

Cor. 5.—In a series of odd numbers, beginning at 1, a 1, 3, 5, &c., $s = n^2$; for the sum of the extremes is doubl the number of terms.

1. Required the 12th term of the series 5, 8, 11, &c.

Here $n = 12$, $a = 5$, $d = 3$; therefore $y = 5 + 11 \times 3 = 3$

2. Required the 7th term of the series 182, 178, 174, &c.

Here $n = 7$, $y = 182$, $d = 4$; therefore $a = 182 - 6 \times$ $= 158$.

3. Required the sum of 12 terms of the series 3, 8, 13, &

Here $a = 3$, $d = 5$, $n = 12$, $y = 3 + 11 \times 5 = 58$, a $s = (58 + 3)6 = 366$.

4. Required the sum of 14 terms of the series 89, 85, 81, &

Here $a = 89 - 13 \times 4 = 37$, and $s = (89 + 37)7 = 882$

From these propositions any two of the five things me tioned may be found, if the other three be given. The theore for finding them are expressed in the following Table.

Given.	Least $=a$.	Greatest $=y$.	Difference $=d$.	Number of Terms $=n$.	Sum $=s$.
$a,\ y,\ n$			$\frac{y-a}{n-1}$		$\frac{1}{2}n(y+a)$
$a,\ d,\ n$		$a+(n-1)d$			$\frac{1}{2}n(2a+(n-1)d)$
$a,\ n,\ s$		$\frac{2s}{n}-a$	$\frac{s-an}{\frac{1}{2}n(n-1)}$		
$y,\ n,\ s$	$\frac{2s}{n}-y$		$\frac{ny-s}{\frac{1}{2}n(n-1)}$		
$y,\ n,\ d$	$y-(n-1)d$				$\frac{1}{2}n(2y-(n-1)d)$
$d,\ n,\ s$	$\frac{s}{n}-\frac{n-1}{2}d$	$\frac{s}{n}+\frac{n-1}{2}d$			
$a,\ y,\ s$			$\frac{y^2-a^2}{2s-y-a}$	$\frac{2s}{y+a}$	
$a,\ y,\ d$				$\frac{y-a}{d}+1$	$\frac{(y+a)\times(y+d-a)}{2d}$
$a,\ d,\ s$		$\frac{(8ds+\overline{2a-d}\vert^{2})^{\frac{1}{2}}-d}{2}$		$\frac{(8ds+\overline{2a-d}\vert^{2})^{\frac{1}{2}}-2a+d}{2d}$	
$y,\ d,\ s$	$\frac{d\pm(\overline{2y+d}\vert^{2}-8ds)^{\frac{1}{2}}}{2}$			$\frac{2y+d\pm(\overline{2y+d}\vert^{2}-8ds)^{\frac{1}{2}}}{2d}$	

USE OF THE TABLE.

1. Let the least term be 7, the common difference 2, and the sum of the series 567. Required the greatest, and the number of terms.

$\surd(567\times 8\times 2+\overline{14-2}|^2)=\surd(9072+144)=\surd 9216=96$, and $\frac{96-2}{2}=47$, the greatest term; and $\frac{96-14+2}{2\times 2}=21$, the number of terms.

2. Given the least term 5, the number of terms 30, and the sum of the series 1455; to find the greatest term and the common difference.

$\frac{1455\times 2}{30}-5=92$ the greatest, $\frac{1455-5\times 30}{15\times 29}=3$ the difference.

3. Given the common difference 4, the number of terms 20, and the sum of the series 1240; to find the least and greatest terms.

$$\frac{1240}{20}\pm\frac{19}{2}\times 4=62\pm 38=100 \text{ and } 24.$$

GEOMETRICAL PROGRESSION.

If a be the least term of a geometrical progression, y the greatest, r the common multiplier or divisor, called the common ratio, and n the number of terms, such a series, if ascending, may be expressed thus, a, ar, ar^2, ar^5, &c., or if descending, thus, y, $\frac{y}{r}$, $\frac{y}{r^2}$, $\frac{y}{r^3}$, &c.; where the exponent of r is one less than the number of the term.

PROP. I.—The greatest term of a geometrical progression is equal to the least term, multiplied by that power of the common ratio, of which the exponent is the number of terms wanting one.

For in the nth term, the exponent of r is $n-1$.

Therefore $y=ar^{n-1}$, and $a=\frac{y}{r^{n-1}}$.

Hence if $a=1$, $y=r^{n-1}$.

Required the 8th term of the series 2, 6, 18, &c.
Here $a=2$, $r=3$, $n=8$; therefore $2\times 3^7=4374$.

PROP. II.—The product of the extremes is equal to the product of any two terms equally distant from the extremes.

For $a\times y=ar\times\frac{y}{r}=ar^2\times\frac{y}{r^2}$, &c.

Cor. 1.—The square of any term is equal to the product of any two terms equally distant from it.

Cor. 2.—If there be four terms, the product of the means, divided by either extreme, gives the other; and if there be three terms, the square of the mean, divided by either extreme, gives the other.

1. Required a third proportional to 85 and 425. Ans. 2125.
2. . . . . . a fourth proportional to 18, 54, 162. . 486.

Prop. III.—If the sum of a geometrical progression be multiplied by the common ratio, and the series be subtracted from the product, the remainder will be equal to the excess of the product of the highest term by the ratio, above the least term.

For the whole series, except the least term, will be included in the product. Thus, if $a+ar+ar^2$, &c. $+\frac{y}{r^2}+\frac{y}{r}+y=s$ be multiplied by r, it becomes $ar+ar^2$, &c. $+\frac{y}{r}+y+yr=sr$; and subtracting the original series, we obtain $yr-a=sr-s$.

Whence $s=\frac{yr-a}{r-1}=\frac{a(r^n-1)}{r-1}$.*

Cor. 1.—The difference between any two adjacent terms is equal to the less multiplied by the ratio, wanting one.

Thus, $ar^3-ar^2=ar^2\times(r-1)$. Wherefore, if the difference of the extremes be multiplied by the greatest term but one, and divided by the difference between the two greatest terms, the quotient will be the sum of all the terms except the

* In this formula r may represent any quantity, integral or fractional, except unity. If $r=1$, there could be no progression; for every power of 1 is 1, and therefore the formula would be $\frac{a(1-1)}{1-1}=\frac{a\times0}{0}$, a very improper expression. When a is multiplied by a quantity less than 1, the product is less than the multiplicand; and the less that the multiplier is taken, the less will the product be; so that $a\times0=0$, or less than any quantity. Again, when a is divided by a quantity less than 1, the quotient is greater than a; and the less that the divisor is taken, the greater will the quotient be: therefore $\frac{a}{0}$ will be infinitely great, or greater than any quantity. To avoid this absurdity, divide first by the denominator, and then affix values to the quantities. If ar^n-a be divided by $r-1$, the quotient is $ar^{n-1}+ar^{n-2}+ar^{n-3}$, &c.; and if $r=1$, it will be $a(1+1+1+1$, &c.$)=na$, which, though not a geometrical progression, is a determined quantity. In like manner $\frac{x^2-a^2}{x-a}$ would be $\frac{0}{0}$, if x were $=a$; but if we divide first, the quotient will be $x+a$, which is $=2a$, when $x=a$. And many other cases may occur like these.

greatest. For the divisor is the product of the multiplier by $r-1$.

Cor. 2.—If the common ratio be 2, the difference of the extremes is the sum of all the terms except the greatest.

Cor. 3.—If a descending series be interminate, the least term may be considered $=0$, and the sum $=\frac{yr}{r-1}$.

1. Required the 8th term of the series 4, 8, 16, &c.

$$4\times 2^7=4\times 128=512.$$

2. Required the sum of 12 terms of the series 1, 3, 9, 27, &c. $\frac{3^{12}-1}{3-1}=\frac{531441-1}{2}=265720.$

3. Required the sum of 8 terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. $\frac{1-\overline{\tfrac{1}{3}}|^{8}}{1-\frac{1}{3}}=\left(1-\frac{1}{6561}\right)\times\frac{3}{2}=\frac{6560}{6561}\times\frac{3}{2}=\frac{3280}{2187}.$

4. Given the extremes a and y, and the sum of the series s, to find the common ratio and the number of terms.

Ans. $r=\frac{s-a}{s-y}$. Having found r, $r^{n-1}=\frac{y}{a}$. And in logarithms, where R, Y, and A represent the logarithms of r, y, and a, $(n-1)\mathrm{R}=\mathrm{Y}-\mathrm{A}$, and $n=\frac{\mathrm{Y}-\mathrm{A}+\mathrm{R}}{\mathrm{R}}$.

QUESTIONS.

1. To find four numbers in arithmetical progression, such, that their sum shall be 56, and the sum of their squares 864. Let the series be x, $x+y$, $x+2y$, $x+3y$, their sum $4x+6y=56$, or $2x+3y=28$, and the sum of their squares $4x^2+12xy+14y^2=864$, from which subtract $\overline{2x+3y}|^2=28^2$, or $4x^2+12xy+9y^2=784$; the remainder gives $5y^2=80$, or $y=4$, and $x=8$; and the numbers are 8, 12, 16, 20.

2. To find three numbers in arithmetical progression, such, that their sum shall be 9, and the sum of their cubes 153. Let the numbers be $x-y$, x, $x+y$, their sum $3x=9$, the sum of their cubes $3x^3+6xy^2=153$.

Ans. The numbers are, 1, 3, 5.

3. To find three numbers in arithmetical progression, such, that their sum shall be 15, and the sum of the squares of the extremes 58. The numbers, $x-y$, x, $x+y$. Ans. 3, 5, 7.

4. To find four numbers in arithmetical progression, such, that the sum of the extremes shall be 8, and the product of the means 15. Ans. 1, 3, 5, 7.

5. To find four numbers in arithmetical progression, such,

that the sum of the squares of the means shall be 52, and the sum of the squares of the extremes 68. Ans. 2, 4, 6, 8.

6. A traveller goes 9 miles a-day: after 7 days another sets out after him, and travels 4 miles the first day, 5 miles the second, 6 miles the third, and so on. In what time will he overtake the first?

$$\frac{8+x-1}{2}x=(x+7)9. \qquad \text{Ans. 18 days.}$$

7. To find three numbers in geometrical progression, such, that their sum shall be 7, and the sum of their squares 21. Let x, y, z, be the numbers.

$$xz=y^2,\ x+y+z=7,\ x^2+y^2+z^2=21. \qquad \text{Ans. } 1, 2, 4.$$

8. To find four numbers in geometrical progression, such, that their sum shall be 30, and that the greatest shall be equal to the sum of the means multiplied by $1\frac{1}{3}$.

x, xy, xy^2, xy^3, the numbers. Ans. 2, 4, 8, 16.

9. To find three numbers in geometrical progression, such, that their product shall be 64, and the sum of their cubes 584. x, xy, xy^2, the numbers.

$$x^3y^3=64,\ x^3\times(1+y^3+y^6)=584. \qquad \text{Ans. } 2, 4, 8.$$

10. To find three numbers in geometrical progression, such, that the sum of the first and third shall be 52, and their product 100. Ans. 2, 10, 50.

11. To find two mean proportionals between 4 and 256. 4, $4x$, $4x^2$, 256, are the proportionals.

Ans. $x^3=\frac{256}{4}=64$, $x=4$, the numbers 4, 16, 64, 256.

12. Given the sum of the squares a, of three numbers in arithmetical progression, and the excess of the square of the mean above the product of the extremes b; to find the numbers.

Ans. Comm. diff. $\sqrt{b}$, mean $\sqrt{\frac{a-2b}{3}}$.

13. Given the product of the extremes a, and the product of the means b, of four numbers in arithmetical progression; to find the numbers.

Ans. Com. diff. $\sqrt{\frac{b-a}{2}}$, least $\frac{1}{2}\left(\sqrt{\frac{9b-a}{2}}-3\sqrt{\frac{b-a}{2}}\right)$.

14. Given the number of terms n, of an arithmetical pro-

gression, their sum a, and the sum of their squares b; to find the terms.

$$\text{Ans. Com. diff. } \left(\frac{12nb-12a^2}{n^2(n^2-1)}\right)^{\frac{1}{2}}.$$

$$\text{Least } \frac{a}{n}-\frac{n+1}{2}\left(\frac{12nb-12a^2}{n^2(n^2-1)}\right)^{\frac{1}{2}}.$$

15. Suppose two travellers set out at the same time from two places of which the distance is given, p. The miles travelled by the first per day form a decreasing arithmetical progression, of which the first term is given, a, and the common difference d. Those travelled by the second form an increasing series, of which the first term is b, and the common difference c. In what time will they meet?

Let $a+b=m$, and $c-d=n$.

$$\text{Ans. } \frac{1}{2}-\frac{m}{n}\pm\sqrt{\left(\frac{2p}{n}+\left(\frac{m}{n}-\frac{1}{2}\right)^2\right)}, \text{ or (if } n=0); \frac{p}{m}.$$

16. Given the sum s of five numbers in geometrical progression, and the sum of their squares a; to find the numbers. Suppose $v=$ sum of the first and third, then $v=\frac{s}{2}-\frac{a}{2s}$, and the second $=\sqrt{\left(v^2+\left(\frac{s-v}{2}\right)^2\right)}-\frac{s-v}{2}$.

INTEREST AND ANNUITIES.

In Simple Interest, the interest is computed on the principal only. Let $p=$ principal or money lent, $t=$ time, $r=$ rate or interest of £1 for the time one, $i=$ interest for the whole time, $a=$ amount or sum of principal and interest; then $rt=$ interest of £1 for the time t, and $1+rt$ the amount of £1, and $p\times(1+rt)=p+prt=p+i=a$ the amount of the whole; from which equations the value of any of the quantities concerned may be found in terms of the others.

In Compound Interest, the interest at each term of payment is added to the principal, and the amount is the principal for the next term. Let $R=1+r$ the amount of £1 for the first term, it will be the principal for the next term, and the interest upon it will be Rr, and the amount $Rr+R=R(r+1)=R^2$ will be the principal for the next term. In like manner we find that the amounts at the end of the following terms will be R^3, R^4, &c.; and at the end of the time t it will be R^t, and for the principal p it will be pR^t the amount, and the interest will be $pR^t-p=i=a-p$; from which equations any of the quantities may be expressed in terms of the rest.

Of Annuities.—If m = principal, which yields £1 of annual interest at the given rate, then $mR^t - m$ = interest of this principal for the time t, which will therefore be the amount of an annuity of £1 for that time. But $m = \frac{1}{r}$, and therefore the amount will be $\frac{R^t - 1}{r}$; and for any annuity n, it will be $\frac{nR^t - n}{r} = a$. And if p be equal to the present value of this annuity, then $\frac{nR^t - n}{r} = pR^t$, and $p = \frac{n - \frac{n}{R^t}}{r}$, where $\frac{1}{R^t}$ is the present worth of £1.

Of Reversions.—When the annuity does not commence till some time after this, it is said to be in reversion. The amount, if it were to commence just now, would be $n \times \frac{R^t - 1}{r}$; but if it commence s years after this, it will be $\frac{n}{R^s} \times \frac{R^t - 1}{r} = a$, and the present worth $p = \frac{n}{R^s} \times \frac{1 - \frac{1}{R^t}}{r}$. From these equations any of the quantities may be expressed in terms of the others.

In a Freehold Estate, the value $y = \frac{1}{r}$ when the rent is £1, and it commences just now; and $\frac{1}{R^s r}$ is its value, when it does not commence till s years after this, y is called the year's purchase or perpetuity, and $ay = v$ the value of the estate, of which the rent is a, and $\frac{ay}{R^s}$ is the value in reversion.

Annuities on Lives.—Adopting Mr De Moivre's hypothesis, that of a certain number born at one time, one dies every year until the whole is extinct, a supposition which agrees nearly with observation, for ages between 10 and 60. An annuity of £1 for a given life will be the sum of the series $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}$, &c., continued to $\frac{n-n}{nr^n}$, where n is the complement of the age, or what it wants of the age at which the oldest dies, which he supposed to be 86, and r the amount of £1 for a year. This sum is $\frac{\left(n-1+\frac{1}{r^n}\right)r - n}{n \times \overline{r-1}|^2} = \frac{n-1-q}{n(r-1)}$, supposing q to be the present worth of an annuity of £1 for $n-1$ years.

Again, the value of an annuity for two joint lives, of which the complements are n and m, (the greatest m)

will be $\frac{n-1}{n}\times\frac{m-1}{mr}+\frac{n-2}{n}\times\frac{m-2}{mr^2}+\frac{n-3}{n}\times\frac{m-3}{mr^3}$, &c. continued to $\frac{n-n}{n}\times\frac{m-n}{mr^n}$, of which the sum is $\frac{1}{r-1}+\frac{(m-n)\frac{1}{r^n}-(m+n)}{mn}\times\frac{r}{\overline{r-1}|^2}+\frac{\left(1-\frac{1}{r^n}\right)(r+1)r}{mn\times\overline{r-1}|^3}$; or if $s=$ value of the oldest life, the value of the two lives is $\frac{(n-1)p-s\times(2p+1-(m-n))}{m}$, where $p=$ perpetuity.

If a question occur which involves both interest and annuities, an equation may be found answering to it by comparing with one another the values of the quantities found separately.

1. What will £1000 amount to in 10 years, at 5 per cent. compound interest? Ans. £1628, 16s.

2. What principal will, in 15 years, amount to £2000, at 4 per cent. compound interest? Ans. £1110, 12s.

3. In what time will £200 amount to £318, 16s., at 6 per cent. compound interest? Ans. 8 years.

4. In what time will a sum of money double itself, at 4 per cent. compound interest? $\overline{1{\cdot}04}|^t=2$. Ans. 17·6 years.

5. Required the amount of £20 annuity for 40 years, at 5 per cent. Ans. £2536, 16s.

6. What annuity will, in 7 years, amount to £79, at 4 per cent.? Ans. £10.

7. What is the value of an annuity of £20, for a life of 54 years of age, at 4 per cent.? Ans. £209·56.

8. What is the value of an annuity of £20, during the joint lives of two persons, whose ages are 35 and 25 years, at 4 per cent.? Ans. £221·9.

9. When 12 years of a lease of 21 years were expired, a renewal for the same term was granted for £1000. Eight years of that lease are now expired, and it is required what sum should be paid for a corresponding renewal of the lease, reckoning 5 per cent. compound interest.

From the first transaction, find the annuity $n=$ £175·029, and from it find p, the present worth of the annuity in reversion $=$ £599·93.

OF SERIES.

A Series is an assemblage of terms, which continually increase or decrease according to a certain law, as the arithmetical and geometrical series spoken of before.

A Converging Series is that of which the terms continually decrease, and a Diverging Series is one of which the terms continually increase.

Series are obtained by division, by the extraction of roots, and by various other operations.

Thus, $\frac{ax}{a-x}=x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}$, &c., where the exponents increase by one.

Also, $\sqrt{a^2+x^2}=a+\frac{x^2}{2a}-\frac{x^4}{2\cdot4a^3}+\frac{3x^6}{2\cdot4\cdot6a^5}-\frac{3\cdot5x^8}{2\cdot4\cdot6\cdot8a^7}+\frac{3\cdot5\cdot7x^{10}}{2\cdot4\cdot6\cdot8\cdot10a^9}$, &c.

Lemma. If the series $(a+b)x+(c+d)x^2+(e+f)x^3$, &c. continued indefinitely, be always $=$ nothing, whatever be the value of x, then the coefficient of any one power of x is $=0$; that is, $a+b=0$, $c+d=0$, &c. For if the equation be divided by x, then $a+b+(c+d)x+(e+f)x^2=0$. Here let $x=0$, then $a+b=0$; therefore $(c+d)x+(e+f)x^2=0$, whatever be the value of x; and proceeding in the same way we find $c+d=0$, and so on.

A GENERAL METHOD OF FORMING SERIES.

Assume a series with unknown but constant coefficients, and having the indices of x increasing or decreasing, in the same way as if the operation were performed at large; then make this series equal to the given quantity, and having cleared the equation of surds and fractions, bring all the terms to one side, so as to make the equation $=0$; after which make the sum of the coefficients of each power of $x=0$, which will give as many equations as there are unknown coefficients; and therefore the values of these coefficients may be found, and substituted for them in the assumed equation.

1. Required a series $=\frac{a}{b+x}$. Assumed $A+Bx+Cx^2+Dx^3$, &c. $=\frac{a}{b+x}$; and by multiplying by $b+x$, and transposing, we get $Ab-a+(Bb+A)x+(Cb+B)x^2$, &c. $=0$, an equation which must be true, whatever be the value of x. Therefore $Ab-a=0$, $Bb+A=0$, $Cb+B=0$, &c. whence $A=+\frac{a}{b}$, $B=\frac{-a}{b^2}$, $C=\frac{+a}{b^3}$, &c.; and these values, substi-

tuted for A, B, C, &c. in the assumed equation, give $\frac{a}{b+x}=\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}$, &c.

2. Let it be $\frac{a^2}{a^2+2ax-x^2}$. Ans. $1-\frac{2x}{a}+\frac{5x^2}{a^2}-\frac{12x^3}{a^3}$, &c.

3. Let it be $\sqrt{a^2-x^2}$. . $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}$, &c.

4. Let it be $\frac{1+2x}{1-x-x^2}$.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5$, &c.

This is a recurring series, in which each of the coefficients after the second is the sum of the two preceding ones.

SUMMATION OF SERIES.

To sum a series is to find a terminated expression equal to the interminate series.

I. Let $a+b+c+d$, &c. be any series; subtract each of the terms from the one following it, and the differences will be $-a+b$, $-b+c$, $-c+d$, &c. This is called the first order of differences. Subtract each of these from the following for a second order of differences, viz. $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. Subtract these again to get another order of differences, and so on.

II. Let d' be the first term of the first order of differences, d'' the first term of the second order, and d''', d'''', d''''', &c. the first terms of the following orders; then $d'=-a+b$, $d''=a-2b+c$, $d'''=-a+3b-3c+d$, $d''''=a-4b+6c-4d+e$, &c., from which we infer that d^{n}* $=\pm a\mp nb\pm n.\frac{n-1}{2}c\mp n.\frac{n-1}{2}.\frac{n-2}{3}d$, &c. to $n+1$ terms; in which series the upper signs are to be used when n is an even number, and the under when it is odd.

Required the first of the fifth differences of the series 6, 9, 17, 35, 63, 99, 148, &c.

$d^{\text{v}}=-6+5\cdot9-5\cdot\frac{4}{2}\cdot17+5\cdot\frac{4}{2}\cdot\frac{3}{3}\cdot35-5\cdot\frac{4}{2}\cdot\frac{3}{3}\cdot\frac{2}{4}\cdot63+5\cdot\frac{4}{2}\cdot\frac{3}{3}\cdot\frac{2}{4}\cdot\frac{1}{5}\cdot99=+3.$

Required the first of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, &c. Ans. -14.

III. Again, since $d'=-a+b$, therefore $b=a+d'$; and in the same manner we get $c=a+2d'+d''$, $d=a+3d'+3d''+d'''$, $e=a+4d'+6d''+4d'''+d''''$, &c., and therefore

* Here n is not the exponent of a power, but the index of the order of differences.

the nth term of the series $= a+(n-1)d'+\frac{n-1}{1}\cdot\frac{n-2}{2}d''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}d'''$, &c. $=z$.

Required the 7th term of the series 3, 5, 8, 12, 17, &c.

Here $d'=2$, $d''=1$, $d'''=0$, and the 7th term, or $z=3+6\cdot2+6\cdot\frac{5}{2}\cdot1=3+12+15=30$.

What is the 9th term of the series 1, 5, 15, 35, 70, &c.?

Ans. 495.

IV. Also, if these values of a, b, c, &c. be added, we obtain $2a+d'=a+b$, $a+b+c=3a+3d'+d''$, $a+b+c+d=4a+6d'+4d''+d'''$, &c. Whence we conclude, that the sum of n terms $s=na+n\cdot\frac{n-1}{2}d'+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}d''$, &c.

If the differences come at last to be equal, these two last series will terminate, otherwise they will be interminate.

1. Required the sum of 8 terms of the series 2, 5, 10, 17, &c.

Here $d'=3$, $d''=2$, $d'''=0$; therefore $s=8\cdot2+8\cdot\frac{7}{2}\cdot3+8\cdot\frac{7}{2}\cdot\frac{6}{3}\cdot2=16+84+112=212$.

2. What is the sum of 12 terms of the series 21, 56, 126, 252, 462, 792, &c.?

Ans. 27125.

3. Required an expression for the sum of n terms of the fourth order of figurate numbers, 1, 4, 10, 20, 35, &c.

Here $d'=3$, $d''=3$, $d'''=1$, and $d''''=0$, and $s=n+n\cdot\frac{n-1}{2}\cdot3+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot3+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot1$, which, reduced, gives $s=n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$. Thus 12 terms are $=\frac{12}{1}\cdot\frac{13}{2}\cdot\frac{14}{3}\cdot\frac{15}{4}=1365$.

The number of factors in the formula, and the order of differences which become $=0$, are the same with the order of the figurates.

4. What is the sum of n terms of the squares $(m\pm a)^2+(m\pm 2a)^2+(m\pm 3a)^2$, &c. $+(m\pm na)^2$?

Ans. $nm^2\pm n\cdot\frac{n+1}{2}\cdot2ma+n\cdot\frac{n+1}{2}\cdot\frac{2n+1}{3}\cdot a^2$.

5. Required the sum of 12 terms of the series $3^2+5^2+7^2+9^2$, &c.

Here $m=1$, $a=2$, and $n=12$; therefore the sum is $12\times1+12\cdot13\cdot2+12\cdot\frac{13}{2}\cdot\frac{25}{3}\cdot4=2924$.

6. Required the sum of n terms of the series of cubes $(m \pm a)^3 + (m \pm 2a)^3 + (m \pm 3a)^3$, &c. $+ (m \pm na)^3$.

Ans. $nm^3 \pm n \cdot \frac{n+1}{2} \cdot 3m^2a + n \cdot \frac{n+1}{1} \cdot \frac{2n+1}{2} \cdot ma^2 \pm n^2\left(\frac{n+1}{2}\right)^2 a^3$.

7. Required the sum of nine terms of the series $3^3 + 6^3 + 9^3 + 12^3$, &c.

Here $m=0$, $a=3$, and $n=9$; consequently the three first terms of the formula are $=0$, and the sum is $n^2\left(\frac{n+1}{2}\right)^2 a^3 = 54675$.

8. Required the sum of a series of products $pq + (p-1) \times (q-1) + (p-2) \times (q-2) + (p-3) \times (q-3)$, &c.

Ans. $\frac{3pq^2 + 3pq - q^3 + q}{6}$. If the number of terms n, be less than q, the answer will be $npq - n \cdot \frac{n-1}{2}(p+q) + n \cdot \frac{n-1}{2} \cdot \frac{2n-1}{3}$.

REVERSION OF SERIES.

When an equation is given of this form, $x = az + bz^2 + cz^3 + dz^4$, &c., and it is required to find z in terms of x, this is called the Reversion of the Series.

Assume the equation $z = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c., and substituting this series and its powers instead of z and its powers in the given equation, make the coefficients of the like powers of x each $=0$, and they will give equations for finding the values of A, B, C, D, &c. This will be best understood from an example.

Let $x = v + \frac{1}{6}v^3 + \frac{3}{40}v^5 + \frac{15}{336}v^7 + \frac{105}{3456}v^9 +$ &c., and let it be required to find v in terms of x.

Here the assumed equation is $v = Ax + Bx^3 + Cx^5 + Dx^7 + Ex^9$, &c. Therefore,

$$\frac{1}{6}v^3 = +\frac{1}{6}x^3 + \frac{3}{6}Bx^5 + \frac{B^2+C}{2}x^7 + \left(\frac{1}{2}D + AB + A^3\right)x^9, \text{ \&c.}$$

$$\frac{3}{40}v^5 = \quad + \frac{3}{40}x^5 + \frac{15}{40}Bx^7 + \left(\frac{3}{4}B^2 + \frac{3}{8}C\right)x^9, \text{ \&c.}$$

$$\frac{15}{336}v^7 = \quad + \frac{15}{336}x^7 + \frac{5}{16}Bx^9, \text{ \&c.}$$

$$\frac{105}{3456}v^9 = \quad + \frac{105}{3456}x^9, \text{ \&c.}$$

And equating the coefficients of the like powers of x, we have

$B+\frac{1}{6}=0$ or $B=-\frac{1}{6}$, $C+\frac{3}{6}B+\frac{3}{40}=0$ or $C=+\frac{1}{120}$, $D+\frac{1}{2}B^2+\frac{1}{2}C+\frac{3}{8}B+\frac{5}{112}=0$ or $D=-\frac{1}{5040}$, &c. Therefore $v=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7$, &c. $=x-\frac{x^3}{2\cdot3}+\frac{x^5}{2\cdot3\cdot4\cdot5}-\frac{x^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+$ &c., where the law of continuation is evident.

REVERT THE FOLLOWING SERIES:

1. $x=y-y^2+y^3-y^4$, &c.

Ans. $y=x+x^2+x^3+x^4$, &c.

2. $x=y+\frac{1}{2}y^2+\frac{1}{3}y^3+\frac{1}{4}y^4$, &c.

Ans. $y=x-\frac{x^2}{2}+\frac{x^3}{2\cdot3}-\frac{x^4}{2\cdot3\cdot4}+\frac{x^5}{2\cdot3\cdot4\cdot5}$, &c.

3. $x=\frac{y}{a}-\frac{y^2}{2a^2}+\frac{y^3}{3a^3}-\frac{y^4}{4a^4}$, &c.

Ans. $y=a\times\left(x+\frac{x^2}{2}+\frac{x^3}{2\cdot3}+\frac{x^4}{2\cdot3\cdot4}\right)$, &c.

4. $x=y-\frac{y^3}{2\cdot3a^2}+\frac{y^5}{2\cdot3\cdot4a^4}-\frac{y^7}{2\cdot3\cdot4\cdot5\cdot6a^6}$, &c.

Ans. $y=x+\frac{x^3}{2\cdot3a^2}+\frac{x^5}{2\cdot3\cdot4a^4}+\frac{x^7}{2\cdot3\cdot4\cdot5\cdot6a^6}$, &c.

5. $x=\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}$, &c. put $v=2x$.

Ans. $y=v^{\frac{1}{2}}-\frac{v}{3}+\frac{v^{\frac{3}{2}}}{36}-\frac{v^2}{170}$, &c.

6. $x=y^{-\frac{1}{2}}-\frac{y^{\frac{1}{2}}}{2}-\frac{y^{\frac{3}{2}}}{8}-\frac{y^{\frac{5}{2}}}{16}-\frac{y^{\frac{7}{2}}}{121}$, &c.

Ans. $y=x^{-2}-x^{-4}+x^{-6}-x^{-8}$, &c.

LOGARITHMS.

LOGARITHMS are a set of artificial numbers, so adapted to the natural numbers, that, by their aid, addition supplies the place of multiplication; that is, the sum of the logarithms of two or more numbers is equal to the logarithm of their product. Therefore, if A, B, C, &c. represent the logarithms of a, b, c, &c., then, according to their nature, log. $ab=A+B$, log. $\frac{a}{b}=A-B$, log. $a^n=nA$, log. $a^{\frac{1}{n}}=\frac{A}{n}$; whence log. $\frac{ax^n}{z^m}=A+nX-mZ$, log. $(a^2-b^2)^{\frac{1}{2}}=\frac{1}{2}$ log. $(a+b)+\frac{1}{2}$ log. $(a-b)$. Log. $a^3\times a^{\frac{3}{4}}=\frac{15A}{4}$.

TO CALCULATE LOGARITHMS BY SERIES.

Let the logarithm of $1+x$ be required. Let $1+z=\overline{1+x}^n$. Assume $\log.(1+x)=Ax+Bx^2+Cx^3$,* &c., and, by a similar assumption, $\log.\ 1+z=Az+Bz^2+Cz^3$, &c. But $z=nx+n\cdot\frac{n-1}{2}x^2$, &c., and $\log.\ 1+z=n\times\log.\ 1+x$ $=nAx+nBx^2+nCx^3$, &c.: therefore, by substituting the value of z in the first expression of $\log.\ 1+z$, and making it equal to the other, and then equating the coefficients, we get $A=A$, $B=-\frac{1}{2}A$, $C=+\frac{1}{3}A$, $D=-\frac{1}{4}A$; and therefore $\log.\ 1+x=A\times(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$, &c.) Now $\log.\ a\times(1+x)=\log.\ a+\log.\ 1+x=\log.\ a+A\times(x-\frac{1}{2}x^2+\frac{1}{3}x^3$, &c.); and if $ax=y$, or $x=\frac{y}{a}$, $\log.\ (a+y)=\log.\ a+A\times\left(\frac{y}{a}-\frac{y^2}{2a^2}+\frac{y^3}{3a^3}\text{, \&c.}\right)$; or if $ax=-y$, $\log.\ a-y=\log.\ a-A\times\left(\frac{y}{a}+\frac{y^2}{2a^2}+\frac{y^3}{3a^3}\text{, \&c.}\right)$; and by subtraction, $\log.\ \frac{a+y}{a-y}=\frac{2Ay}{a}\times\left(1+\frac{y^2}{3a^2}+\frac{y^4}{5a^4}+\frac{y^6}{7a^6}\text{, \&c.}\right)$

In these series the quantity A is not determined, but it is a common multiplier of the series, and therefore is constant in the same system. If $A=1$, the system will be that of the natural or hyperbolic logarithms, which were the first invented by Lord Napier. Hence, in any other system, the logarithms may be got by multiplying the natural logarithms by the value of A in that other system. This value is called the *module* of the system.

TO FIND THE NATURAL LOGARITHM OF 10.

$10=\left(\frac{5}{4}\right)^{10}\times\left(\frac{1024}{1000}\right)^3$. Let $\frac{5}{4}=\frac{a+y}{a-y}$, then $a=\frac{9}{2}$, and $y=\frac{1}{2}$, and $\frac{y}{a}=\frac{1}{9}$, and $\frac{y^2}{a^2}=\frac{1}{81}$; therefore $\log.\ \frac{5}{4}=\frac{2}{9}\times$

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* x must enter into every term of the series, that it may become $=0$ when $x=0$; and for the same reason x cannot be in the denominator.

As n may denote any number, and the result is the same whatever its value is, it will be best to take $n=2$; then $\log.\ 1+z=Az+Bz^2+Cz^3$, &c. But $z=x(2+x)$, therefore $\log.\ 1+z=Ax(2+x)+Bx^2(2+x)^2+Cx^3(2+x)^3$, &c. $=2Ax+(A+4B)x^2+(4B+8C)x^3$ $+(B+12C+16D)x^4$, &c.; and it is also $=2Ax+2Bx^2+2Cx^3+2Dx^4$, &c. Here, by equating the coefficients of the same powers of x, we get $A=A$, $B=-\frac{1}{2}A$, &c. as in the text.

$\left(1+\frac{1}{3\cdot81}+\frac{1}{5\cdot81^2}, \&c.\right)=\cdot2231435513$, &c. And making $\frac{1024}{1000}$ $=\frac{a+y}{a-y}$, $a=1012$, and $y=12$, $\frac{y}{a}=\frac{3}{253}$ and $\frac{y^2}{a^2}=\frac{9}{64009}$; therefore log. $\frac{1024}{1000}=2\times\frac{3}{253}\times\left(1+\frac{9}{3\cdot64009}+\frac{9^2}{5\cdot64009^2}, \&c.\right)$ $=\cdot0237165266173$, &c. Wherefore log. $10=10$ log. $\frac{5}{4}+3$ log. $\frac{1024}{1000}=2\cdot302585092994$, &c.

Because the common logarithm of 10 is 1, therefore 1 divided by 2·302585, &c. will give the module of the common logarithms $=\cdot4342944819$, &c.

Hence the natural logarithm multiplied by ·43429448, &c. will give the common logarithm; and the common logarithm multiplied by 2·30258, &c. will give the natural logarithm.

To find the number belonging to a natural logarithm. Let $z=\log. 1+x=x-\frac{x^2}{2}+\frac{x^3}{3}$, &c.; and by reversion $x=z+\frac{z^2}{2}+\frac{z^3}{2\cdot3}$, &c., and $1+x=1+z+\frac{z^2}{2}$, &c.

Let $z=1$, then $1+x=1+1\frac{1}{2}+\frac{1}{2\cdot3}+\frac{1}{2\cdot3\cdot4}$, &c. $=2\cdot71828183$, &c., the number of which the natural log. is 1.

Required the common logarithms of the first 12 numbers.

USE OF LOGARITHMS IN EQUATIONS.

1. Let the equation be $a^x=b$; then, in logarithms, $x\mathrm{A}=\mathrm{B}$, and $x=\frac{\mathrm{B}}{\mathrm{A}}$, where the capitals A, B, C, &c. represent the logarithms of the quantities a, b, c, &c.

2. Let $\frac{a^{mx}}{b^{nx-1}}=c$; then $mx\mathrm{A}-(nx-1)\mathrm{B}=\mathrm{C}$, and $x=\frac{\mathrm{C}-\mathrm{B}}{m\mathrm{A}-n\mathrm{B}}$.

3. Let $a^x=\frac{b^{mx-n}}{c^{rx}}$; then $x\mathrm{A}=(mx-n)\mathrm{B}-rx\mathrm{C}$, and $x=\frac{n\mathrm{B}}{m\mathrm{B}-r\mathrm{C}-\mathrm{A}}$.

4. Let $\frac{b^{n-\frac{a}{x}}}{c^{mx}}=d^{x-p}$; then $\left(n-\frac{a}{x}\right)\mathrm{B}-mx\mathrm{C}=(x-p)\mathrm{D}$, whence $x=\frac{p\mathrm{D}+n\mathrm{B}\pm\sqrt{(\overline{p\mathrm{D}+n\mathrm{B}}|^2-4a\mathrm{B}\times(\mathrm{D}+m\mathrm{C}))}}{2\mathrm{D}+2m\mathrm{C}}$.

PROBLEMS.

1. The duties on certain goods amounted to £2460, out of which a discount was allowed of $2\frac{1}{2}$ per cent. upon the sum actually paid for prompt payment. What did the discount amount to? Ans. £60.

2. A merchant discounted two bills at the bank, one of them for £550, payable in 7 months, and the other for £720, payable in 4 months; and he received for the whole £1200. At what rate per cent. per annum was the interest charged? Ans. £13·267 per cent. per annum.

3. The common difference of four numbers in arithmetical progression is 4, and their continual product is 21945. What are the numbers? Ans. 7, 11, 15, 19.

4. The sum of ten numbers in arithmetical progression is 120, and the sum of their cubes is 29160. What are the numbers? Ans. 3, 5, 7, 9, &c.

5. Given the sum of the numbers 0, 1, 2, 3, &c. = 1225; to find the sum of their squares. Ans. 40425.

6. Two persons set out at the same time from two places 462 miles distant, to meet one another. The first goes 1 mile the first day, 2 miles the second day, and so on. The second travels each day the cubes of the number of miles that the first travelled on that day. In what time will they meet? Ans. 6 days.

7. A gentleman sold an estate for the value of the trees upon it above 7 feet circumference, at one pound for the first, two for the second, four for the third, and so on, doubling the price of each successive tree. The value of the estate came to £65535. How many trees of the above description were upon it? Ans. 16 trees.

8. A gentleman had seven children, whose ages differed successively by one year. In giving them new clothes, he determined to bestow as many yards of lace on the trimming of the youngest as he was years old, on the second as many as the sum of the ages of the two youngest, on the third as many as the sum of the ages of the three youngest, and so on; and he agreed with the tailor to pay for making each suit the product in pence of the child's age by the number of yards of lace on his suit. The bill amounted to £7, 10s. 6d. What were the ages of the children? Ans. The youngest 5 years.

9. Required the number of combinations of m things in n things.

Ans. The number of combinations of two in n things is $n\left(\frac{n-1}{2}\right)$; of three, is $n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)$; of four, is

$n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)\left(\frac{n-3}{4}\right)$; of which the number of factors is equal to the number of things in one combination. Therefore the number of combinations of m in n things will be $n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)\left(\frac{n-3}{4}\right)$, &c. to $\frac{n-(m-1)}{m}$.

10. A merchant discounted two bills; one had 6 months to run, and the other 8 months. The value of both came to £308, 6s. 8d., and the discount to £8, 6s. 8d. Had interest been charged upon the bills, it would have come to 4s. 8$\frac{2}{3}$d. more than the discount. Required the value of the bills.

Ans. The bill due at 6 months was £205, and the other £103$\frac{1}{3}$.

11. If £400 be the present value of an annuity to continue 23 years after the expiration of 8 years, what would be its value for 21 years after the expiration of 10 years, interest at 5 per cent.? Ans. £344·9597.

12. A gentleman had 10 different annuities of £100 each; their continuance differed by one year each, and the longest was for 60 years. He sold them all at 5 per cent. compound interest. What money did he receive for them?

Ans. £18653·26.

13. A bookseller purchases a work for £40, and pays for printing 1000 copies of it £15, for paper £20, and for incidents £10. He sells the edition in 10 years at 3s. each copy. How much does he gain per cent. per annum?

Ans. £11..19 per cent. per annum.

14. A person who owes his creditor £320 just now, and £96 more at the end of five years, wishes to pay the whole in one payment. What is the proper time for doing this, according to the true principle of equation of payments, viz. that the simple interest shall be equal to the discount?

Ans. At the end of one year.

15. A usurer lent £186 for a certain time, and gained £31; and by lending £360 at the same rate for another time, he gained £90. The sum of the times they were lent amounted to 20 months. How long time was each sum lent?

Ans. The first 8 months, the other 12 months.

PRACTICAL GEOMETRY.

DEFINITIONS.

1. Geometry treats of magnitude or continued quantity, and of its relation to number.

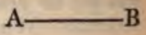
2. A Solid is that which has three dimensions, length, breadth, and thickness.

3. A Surface, or Superficies, is the boundary of a solid, and has only length and breadth.

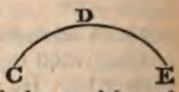
4. A Line is the boundary of a surface, and has only length.

5. A Point is the extremity of a line. It has position, but not magnitude, as A.

6. A Straight Line is uniform on all its sides. It can be exhibited by stretching a hair between two points, as AB.

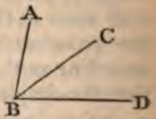


7. A Curve changes continually its direction, or it has unlike sides, a concave and a convex, as CDE.

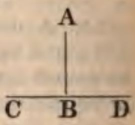


8. An Angle is the measure of the relative position of two straight lines which meet, or it is their inclination to one another.

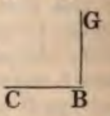
Note. An angle is denoted by three letters, of which the second is at the point where the lines meet, and the other two are upon the containing lines, one on each. Thus the uppermost angle is named ABC, the other CBD, and the whole angle ABD.



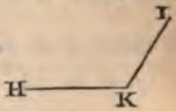
9. A straight line is said to be Perpendicular to another, when it does not incline towards one end more than towards the other. Thus AB is perpendicular to CD.

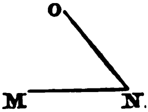


10. A Right Angle is that made by a perpendicular, as CBG.



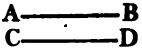
11. An Obtuse Angle is greater than a right angle, as HKI.





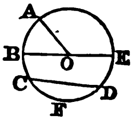
12. An ACUTE ANGLE is less than a right angle, as MNO.

13. A PLANE is a surface with which a straight line will coincide, when drawn between any two points in it.



14. PARALLELS are straight lines in a plane, which never meet, though extended ever so far both ways, as AB and CD.

15. A CIRCLE is a figure contained by a curve ABD, called the *circumference,* which is equally distant from a point O within it, called the *centre.*



16. The RADIUS AO is a straight line, drawn from the centre to the circumference.

17. The DIAMETER BE is a straight line, drawn through the centre O, and terminated both ways at the circumference.

18. A CHORD CD is a straight line joining any two points of the circumference.

19. An ARC BCD is any part of the circumference.

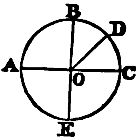
20. A SEMICIRCLE is a portion of the circle cut off by a diameter, as BAE.

21. A SEGMENT is a portion CFD, cut off by a chord CD.

22. A SECTOR is a part cut off by two radii, as AOB.

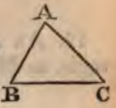
NOTE 1. If the radii contain a right angle, the sector is called a *Quadrant;* and if half a right angle, it is called an *Octant.*

NOTE 2. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees,* and a degree into 60 equal parts, called *minutes,* and a minute into 60 *seconds,* and so on.

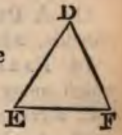


NOTE 3. If two diameters AC, BE, are perpendicular to one another, they divide both the circle and the circumference into four equal parts, and form four right angles at the centre; and if the arc CB of one of these parts be divided into 90 degrees, and radii drawn to the points of division, they will divide the right angle BOC into 90 equal angles, each of which is said to be an angle of one degree, and any angle AOD at the centre is said to consist of as many degrees as the arc AD upon which it stands. The arc AD is called the *measure* of the angle AOD. Hence a right angle AOB contains 90 degrees, an obtuse angle AOD more, and an acute angle COD less than 90 degrees.

23. A Triangle is a figure contained by three straight lines, as ABC.

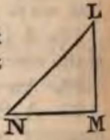


24. An Equilateral Triangle has its three sides equal, as DEF.

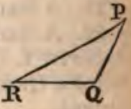


25. An Isosceles Triangle has two of its sides equal, as GHK.

26. A Right-angled Triangle has one right angle, as LMN. The side LN opposite to the right angle is called the *Hypotenuse.*

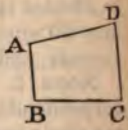


27. An Obtuse-angled Triangle has one obtuse angle, as PQR.

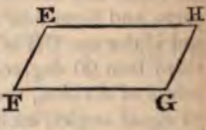


28. All others are called Acute-angled Triangles.

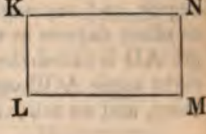
29. A Quadrilateral is a figure bounded by four straight lines, as ABCD.

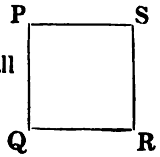


30. A Parallelogram is a quadrilateral, of which the opposite sides are parallel, as EFGH.

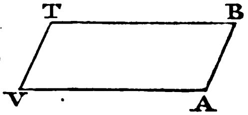


31. A Rectangle is a parallelogram which has right angles, as KLMN.

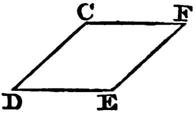




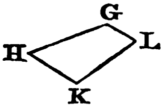
32. A Square is a rectangle which has all its sides equal, as PQRS.



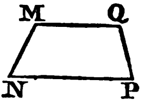
33. A Rhomboid is a parallelogram which has no right angles, as TVAB.



34. A Rhombus is a rhomboid which has all its sides equal, as CDEF.

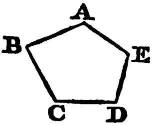


35. A Trapeze, or Trapezium, is a quadrilateral which has not its opposite sides equal, as GHKL.

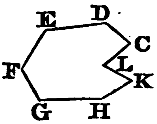


36. A Trapezoid has two sides parallel, but not the other two, as MNPQ.

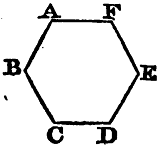
37. A Diagonal is a straight line, which joins two opposite angles of a figure, as MP.



38. A Polygon is a figure contained by more than four straight lines, as ABCDE.



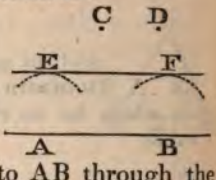
39. A Polygon of five sides is called a *Pentagon;* one of six sides, a *Hexagon;* of seven sides, a *Heptagon;* of eight sides, an *Octagon;* of nine sides, a *Nonagon;* of ten sides, a *Decagon,* &c.



40. A Regular Polygon is that which has all its sides and all its angles equal, as ABCDEF.

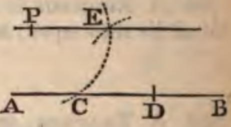
PROBLEMS.

Prob. I. To draw a straight line parallel to AB, and as far from it as the point C is from D.

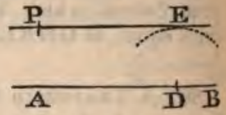


With the distance CD for a radius, describe arcs E and F from the centres A and B, and draw the straight line EF to touch these arcs without cutting them.

Prob. II. To draw a parallel to AB through the point P.

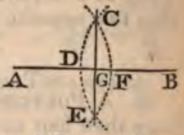


From P, with any sufficient radius, describe an arc cutting AB in C. Lay the radius on AB from C to D, and from D cut the arc again in E, and draw PE.



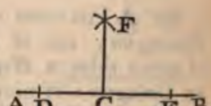
Or, with the nearest distance of P from AB for a radius, describe an arc E, from D, taken as far as possible from P, and draw a line from P to touch the arc E.

Prob. III. To bisect a given straight line AB.



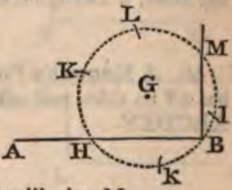
With a radius greater than half the line, describe from B the arc CDE, and from A the arc CFE, cutting the former in C and E. Draw CE cutting AB in G.

Prob. IV. To raise a perpendicular to AB at a given point in it, as C.



With any radius, from C, cut AB in D and E; and with a greater radius describe arcs from D and E, cutting one another in F, and draw CF.

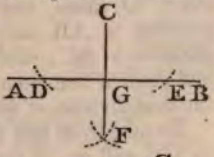
If the perpendicular is to be raised at B, the end of AB,



Place one foot at G, above AB, and extending the other to B, describe a circle cutting AB in H; then lay the radius on the circumference, from H to K, from K to L, and from L to M, and draw BM.

Or a straight line through H and G will give M.

Prob. V. To drop a perpendicular upon AB from the point C above it.

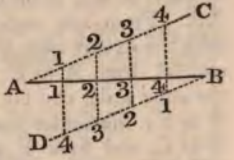


With a sufficient radius, from C cut AB in D and E, and from these points describe arcs on the other side of AB, cutting one another in F, and draw CF, cutting AB in G.

If the point C be above the end of AB,

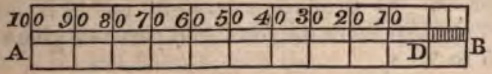
From any point G in AB, with the radius GC, describe the arc CDE; and from any other point H, in AB, with the radius HC, describe the arc CFE, cutting the former in E, and draw CE.

Prob. VI. To divide a straight line AB into any number of equal parts, suppose five.



Through A and B draw any parallels AC and BD, on different sides of AB. Take any convenient distance, and lay it four times (one less than the given number) from A on AC, and from B on BD; then join the first on AC to the fourth on BD, the second on AC to the third on BD, and so on in order, and the joining lines will divide AB into five equal parts.

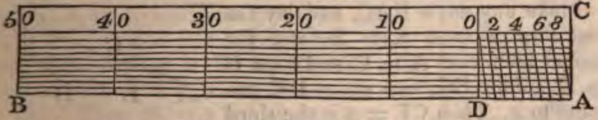
Prob. VII. To make a plane scale, or one of equal parts.



Draw any straight line AB, and take any convenient distance, and lay it eleven times from A to B, and divide the last one BD into 10 equal parts; then each of the large divisions will be 10, and each of the small divisions 1.

For a scale of feet and inches, divide BD into 12 equal parts; then each of the large divisions will be a foot, and each of the small ones an inch.

Prob. VIII. To make a diagonal scale.

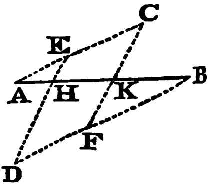


Having drawn AB, and divided it as in the plane scale,

draw AC perpendicular to AB, and on it lay any small distance 10 times, and through the points of division draw parallels to AB, and through the great divisions of AB draw parallels to AC; divide AD and C0 each into 10 equal parts, and draw a line from 0 to the first division of AD, and from the first division of 0C to the second of AD, and so on.

To take from this scale a number consisting of three figures, as 546, call one of the large divisions 100, or take 5 of them, call one of the divisions on 0C 10, or take 4 of them, and for the units reckon one parallel on the diagonal for each unit; or count 6 parallels on the diagonal through 4, and bring the foot on the large 5, along that division to the sixth parallel.

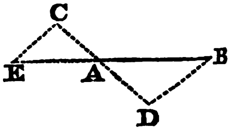
Prob. IX. To divide a straight line AB in any proportions, as of 3, 5, 7.



Draw any parallels AC and BD, through A and B on different sides of AB. From any scale of equal parts take the extent from 0 to 3, and lay it on AC, from A to E. Take 7 from the same scale, and lay it on BD, from B to F; then take 5, and lay it from E to C, and from F to D; and join ED, CF, cutting AB in H and K. AH : HK :: 3 : 5, and HK : KB :: 5 : 7.

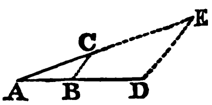
Note. In the same way, AB may be divided similarly to a given divided line.

Prob. X. To produce a straight line AB, so that the whole shall be to the produced part in a given ratio, as of 5 to 2.



Through A draw any straight line AC, lay 2 from A to C, and 5 from C to D towards A. Join BD, and parallel to it draw CE. Then BE : EA :: 5 : 2.

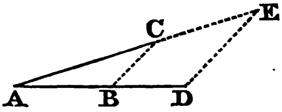
Prob. XI. To find a third line proportional to two given straight lines, as 4 and 6.



Make any angle BAC, and lay the first term 4 from A to B, and the second both from A to C and from B to D. Join BC, and draw DE parallel to it. Then CE = 9 is the third proportional.

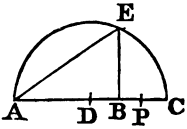
Prob. XII. To find a fourth line proportional to three given ones, as 8, 6, and 12.

Make any angle BAC. Lay the first 8 from A to B, the second 6 from B to D, and the third 12 from A to C. Join BC, and draw DE parallel to it. Then CE is the fourth proportional.



Prob. XIII. To find a mean proportional between two straight lines, as 9 and 4.

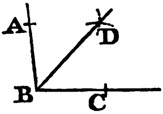
On the same straight line lay AB 9 and BC 4, and bisect AC in D; and with the radius DA describe the semicircle AEC, and draw BE perpendicular to AC. It is the mean proportional, for AB : BE :: BE : BC.



Note. Make AP = AE, then AP or AE is a mean proportional between AC and AB; therefore AC : AB :: $AC^2 : AP^2$.

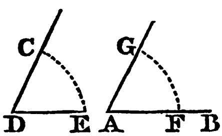
Prob. XIV. To bisect a given angle ABC.

From B, with any radius, cut the sides in A and C. From A describe the arc D, and from C cut it in D, and join BD, the angle ABD = CBD.



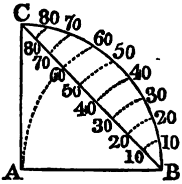
Prob. XV. To make, at A in AB, an angle equal to the angle CDE.

From D, with any radius, cut DC, DE, in C, E; and from A, with the same radius, describe the arc FG, cutting AB in F. Take the extent from C to E, and lay it on the arc from F to G, and draw AG, the angle FAG = CDE.



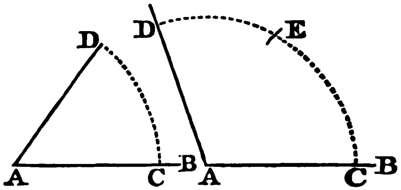
Prob. XVI. To make a scale of chords.

Draw AC perpendicular to AB. From A, with any radius, describe the arc BC, and let it be divided into 90 equal parts, (it is here divided into 9,) and draw BC; and, with one foot in B, transfer the extents to each of the divisions, from the arc to BC. Then BC is a line of chords.



Note. The radius AB is equal to the chord of 60°.

Prob. XVII. To make an angle of any number of degrees, at A in AB.

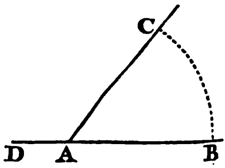


Take 60° from the line of chords, and from A describe an arc, cutting AB in C.

If the given angle do not exceed 90°, as here 54°, take it from the line of chords, and lay it on the arc from C to D, and draw AD; then BAD is the angle required.

If the given number of degrees be greater than 90°, as 112°, take a less number from the chords, and lay it from C to E, and lay the rest from E to D, and draw AD; then BAD is the angle required.

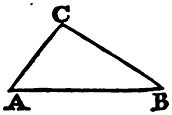
Prob. XVIII. To measure a given angle BAC.



With the chord of 60°, from A describe the arc BC. Take BC, and lay it on the line of chords, and it will show the number of degrees in the angle BAC.

If the extent from B to C be greater than the line of chords, measure part of the arc, and then the rest, and add them. Or produce BA to D, and measure CAD, which, subtracted from 180°, leaves BAC.

Prob. XIX. To make a triangle, of which the three sides are given, viz. 186, 257, and 324 feet.

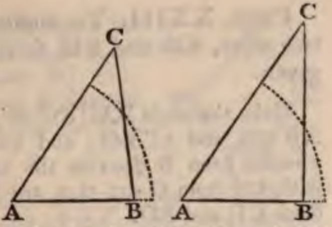


Draw a straight line AB. Take 324 from the diagonal scale, and lay that extent from A to B. Take 186 from the scale, and from A describe an arc, and with 257 for a radius, from B cut that arc in C, and join AC and CB.

Prob. XX. To make a triangle, of which two sides and an angle are given, viz. 256, 384, and 54° $40'$.

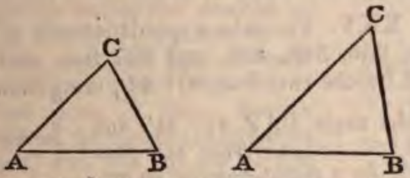
Make the angle BAC 54° 40′, and make AB 256; then, if the given angle be between the given sides, make AC 384, and join BC.

But if one of the sides be opposite to the given angle, with 384 for a radius, from B cut AC in C, and join BC.



Note. If it had been required to make AB 384, and BC 256, the problem would have been impossible; because 256 for a radius would not reach from B to AC. If BC were 340, it would be perpendicular to AC. If BC were greater than 340, but less than 384, it would cut AC in two points, so that two different triangles could then be made with the same things given.

Prob. XXI. To make a triangle, of which two angles 43° 36′, and 57° 44′, and one side 297 feet, are given.

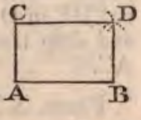


Make the angle BAC 43° 36′, and make AB 297. Then, if the other given angle is to be adjacent to the given side, make ABC 57° 44′; but if it is to be opposite to the given side, add the given angles, and subtract the sum 101° 20′ from 180°. The remainder 78° 40′ is the angle ABC, and then ACB is 57° 44′.

Note. If in either of these problems a right angle is given, it is to be made 90°, or a perpendicular is to be drawn.

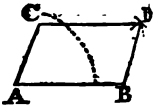
Prob. XXII. To make a rectangle, of which the sides are given; suppose 428 and 246 feet.

Draw AC perpendicular to AB, and make AB 428, and AC 246 feet; and with 246 for a radius, from B describe the arc D; and with 428 for a radius, from C cut that arc in D, and join BD and CD.



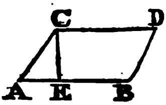
Note. If AC be made equal to AB, the figure will be a square.

PROB. XXIII. To make a parallelogram, of which two sides, 436 and 243 feet, and an angle 67° 30′, are given.



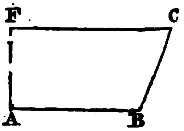
Make the angle BAC 67° 30′, and make AB 436, and AC 243; and with 243 for a radius from B describe the arc D, and with 436 from C cut that arc in D, and draw CD and BD.

PROB. XXIV. To make a parallelogram, of which there are given two sides 421 and 234 feet, and the perpendicular upon one of them, suppose the longest, from the end of the other 196.



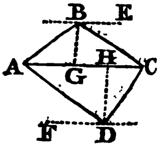
Draw CD parallel to AB, at the distance of 196 feet from it; and with 234 for a radius from A cut CD in C, and make AB and CD each 421, and join AC and BD, and drop the perpendicular CE.

PROB. XXV. To make a quadrilateral, of which all the sides, 256, 348, 436, and 297 feet, and an angle contained by the two first, 87° 44′, are given.



Make the angle BAF 87° 44′, and make AB 256, and AF 348; and from F, with 436 for a radius, describe an arc, and with 297 from B cut that arc in C, and draw FC, CB.

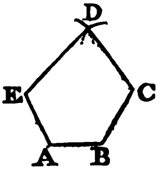
PROB. XXVI. To make a quadrilateral, of which are given two sides 268 and 394, the diagonal from their intersection 473, and the perpendiculars upon it from their extremities 188 and 234 feet.



Make AC 473, and draw parallels to it on different sides at the distances of 188 and 234, as BE and DF. With 268 for a radius from A cut BE in B, and with 394 cut DF in D. Join AB, BC, CD, DA, and drop the perpendiculars BG, DH, on AC.

PROB. XXVII. To make a pentagon, of which all the sides are given, 236, 194, 253, 318, and 372 feet; and two angles, suppose those at the extremities of the second side, 112° and 124°.

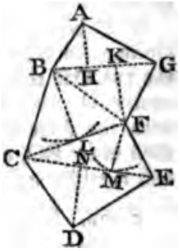
Make AB 194 feet, and at A make the angle BAE 112°, and at B the angle ABC 124°, and make AE 236, and BC 253; then with 318 for a radius from C describe the arc D, and from E with 372 cut it in D, and draw CD and ED.



NOTE. In like manner may any polygon be made, of which all the sides are given, and all the angles except three.

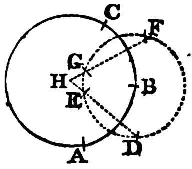
PROB. XXVIII. Given two sides of a figure 234 and 348, the diagonals 438, 385, 452, and 537, and the perpendiculars upon the diagonals from the angles 183, 248, 315, 212, and 274; to make the figure.

First, by Prob. XXVI., make the quadrilateral ABFG, of which AB is 234, BG 438, BF 385, AH 183, and FK 248. From B with the radius 315 describe an arc, and from F draw FC to touch it, and make FC 452, and join BC. From F with 212 make an arc, and draw CE to touch it, and make CE 537. Draw a parallel to CE at the distance of 274 from it, and from C with 348 cut the parallel in D, and join CD, DE, and EF, and draw the perpendiculars BL, FM, and DN.



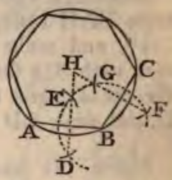
PROB. XXIX. To describe a circle that shall pass through three given points, A, B, C, not in a straight line.

With a radius greater than half the distance of B from A or C describe a circle about B, and with the same radius from A cut the circle in D and E, and from C cut it in F and G. Join DE and FG, meeting one another in H; it is the centre, from which the circle described through A shall pass through B and C.



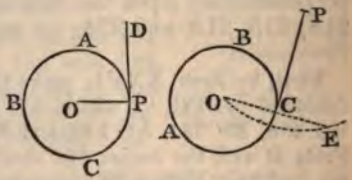
NOTE 1. If ABC be a triangle, a circle may be described about it by this problem. And in the same way, by taking

three points in the circumference, or in any arc of a circle, the centre of that circle may be found.



NOTE 2. The circumference which passes through three of the angular points of a regular polygon passes through all the rest; and therefore a circle may be described about it, or inscribed within it, by this problem.

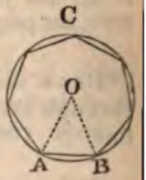
PROB. XXX. To draw a straight line from a given point P, to touch a given circle ABC.



If P be in the circumference, draw PO to the centre, and PD perpendicular to it.

If P be without the circle; from P describe the arc OE through the centre O, and from O, with the diameter of ABC for a radius, cut the arc in E; then draw EO, meeting the circumference in C, and join PC, and it will touch the circle.

PROB. XXXI. To make a regular polygon of a given number of sides in a given circle ABC.



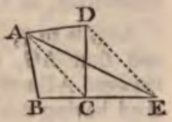
Divide 360° by the number of sides; the quotient is the angle at the centre subtended by one of them. Draw a radius AO, and make the angle AOB equal to the quotient. Join AB, and place straight lines all around the circle equal to AB, and they will form the polygon required.

PROB. XXXII. To make a regular polygon of a given number of sides, upon a given straight line, as AB 365 feet.

Divide 360° by twice the number of sides, and subtract the quotient from 90°, and at A and B make the angles BAO and ABO, each equal to the remainder, and the point O in which the sides meet is the centre of the circle containing the polygon. From O describe a circle through A, and place lines equal to AB all round in it.

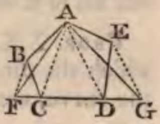
PROB. XXXIII. To make a triangle equal to a given quadrilateral ABCD.

Draw the diagonal AC, and parallel to it, through D, draw DE, meeting BC produced, if necessary, in E, and join AE; then the triangle ABE is equal to the quadrilateral ABCD. For the triangle ACE = ACD.



PROB. XXXIV. To make a triangle equal to a given pentagon ABCDE.

Join AC, and draw BF parallel to it, meeting CD in F, and join AF, and the triangle AFC = ABC; and thus the pentagon is reduced to the quadrilateral AFDE. Let this be reduced as before to the triangle AFG, then AFG = ABCDE.



NOTE. In the same way may any polygon be reduced to a triangle, only the number of operations will increase with the number of the sides of the figure.

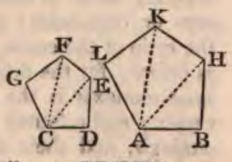
PROB. XXXV. To reduce a triangle ABC to another, which shall have its base in the same straight line with that of the given triangle, and its vertex at a given point P.

Draw PD parallel to BC, meeting AB in D. Join DC, and through A draw AE parallel to DC, and join PB and PE. If DE were joined, the triangle ADC = EDC, and ABC = DBE = PBE.

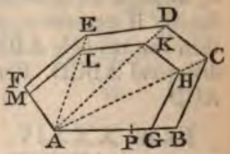
NOTE. By this and the preceding problem, any polygon may be reduced to a triangle, which shall have its vertex at a given point.

PROB. XXXVI. To make a figure upon a given straight line AB, which shall be similar to a given figure CDEFG.

Join CE, CF, to reduce the given figure to triangles. At A make the angle BAH = DCE, HAK = ECF, and KAL = FCG. Also at B make the angle ABH = CDE; at H make AHK = CEF; and at K make AKL = CFG. Then ABHKL is similar to CDEFG.

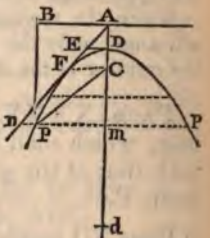


PROB. XXXVII. To make a figure which shall be similar to a given figure ABCDEF, and have a given ratio to it, as that of 7 to 9.



As 9 is to 7, so make AB to AP, and find AG, a mean proportional between AB and AP, by Prob. XIII. And having drawn the diagonals AC, AD, AE, draw GH parallel to BC, meeting AC in H, draw HK parallel to CD, KL parallel to DE, and LM to EF; then the figure AGHKLM is similar to ABCDEF, and has to it the ratio of 7 to 9.

PROB. XXXVIII. To describe a conic section, of which the directrix AB, the focus C, and the ratio of the curve, are given.



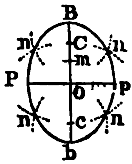
Draw CA perpendicular to AB, and divide it in D, so that CD be to DA in the ratio of the curve, by Prob. IX.

Let CP revolve about C, and at the same time let BP move perpendicular to AB, so that CP : PB always : : CD : DA; then their intersection P will describe the curve.

Or by points. Draw DE parallel to AB, and make it equal to DC, and join AE, and produce it. Draw a great many parallels to AB, meeting AC in *m*, and AE in *n*. Take *mn* on any of them, and from the centre C cut that parallel in P and *p*; these are two points in the curve. In the same manner two points may be found in every parallel, and the curve made to pass through them all.

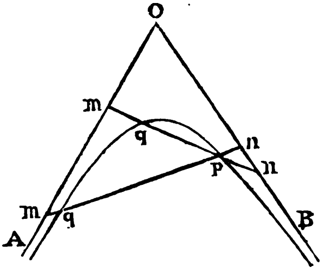
PROB. XXXIX. Given the transverse axis 176, and the conjugate 142, of a hyperbola or ellipse; to describe the curve.

Add the squares of the two semiaxes in the hyperbola, or subtract them in the ellipse, and take the square root of the sum or remainder: this root has to the transverse semiaxis the ratio of the curve, with which the curve may be described as before; for the difference between the root and the transverse semiaxis is the distance of the focus from the principal vertex; and a fourth proportional to the root, the transverse semiaxis, and their difference, will give the distance of the directrix from the principal vertex.



Otherwise, let B*b* and P*p* be the axes, bisecting one another at right angles in the centre O. Lay BP in the hyperbola from O to C and *c*, or lay BO in the ellipse from *p* to C and *c*; then C and *c* are the foci. Take any point *m* in B*b*, produced in the hyperbola, and with the distance B*m* describe two arcs *n*, *n*, from each of the foci C and *c*. Then, with *bm* for a radius, from the foci cut these arcs in *n*, *n*, *n*, *n*; these will be four points of the curve. Take another point *m*, and proceed in the same manner with it to get other four points of the curve, and so on; then draw the curve through all these points.

PROB. XL. Given the asymptotes and a point in the hyperbola; to describe the curve.



Let OA, OB, be the asymptotes, and P the point in the curve.

Through P draw any straight line meeting the asymptotes in *m* and *n*. Make *nq* equal to *m*P, then *q* is a point in the curve. In this way any number of points in the curve may be found, and the curve drawn through them all will be the hyperbola.

LOGARITHMS.

LET a series of numbers in arithmetical progression be adapted to another in geometrical progression, so that the least term of the one correspond with the least of the other and the rest in order thus:

Arith. Prog.	0,	1,	2,	3,	4,	5,	6,	7,	&c.
Geom. Prog.	1,	4,	16,	64,	256,	1024,	4096,	16384,	&c.

And let it be required to multiply any two terms, as 256 and 64 of the geometrical series. This may be done by adding 3 and 4, the corresponding terms of the arithmetical series; for the sum 7 is the term corresponding to 16384 the product.

Thus the use of such an adaptation is manifest; but it is very limited in the present state of the series. In order to extend it, interpose a geometrical mean proportional between every two terms of the geometrical series. This mean is the square root of the product of the adjacent terms. Also interpose an arithmetical mean between every two terms of the arithmetical series, which mean is half the sum of the adjacent terms, and then the number of terms will be doubled, thus:

Ari. Pro.	0,	0·5,	1,	1·5,	2,	2·5,	3,	3·5,	4,	4·5,	5,	5·5,	6,	6·5,	7,	&c.
Geo. Pro.	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	1024,	2048,	4096,	8192,	16384,	&c.

These progressions may be interpolated in the same way by new terms, and the process may be carried on continually, till at length every integer occur in the geometrical series, or a number so near it that the difference may be neglected without error; and then the numbers in the arithmetical series, corresponding to these integers, may be called their *logarithms*.

Hence logarithms are artificial numbers, by the aid of which addition supplies the place of multiplication, and consequently subtraction the place of division.

In forming the common tables of logarithms, the progressions first assumed were,

Arith. Prog.	0,	1,	2,	3,	4,	5,	&c.
Geom. Prog.	1,	10,	100,	1000,	10000,	100000,	&c.

And new terms were interposed continually in the same way as was shown in the preceding series, until the natural numbers occurred in the geometrical series; and then the numbers

in the arithmetical series corresponding to these natural ones were taken to compose the table of logarithms.

Hence the logarithms of all numbers between 1 and 10 are fractions; those of all numbers between 10 and 100 are mixed numbers that have 1 for the integer; those of numbers between 100 and 1000 have 2 for the integer, and so on: that is, the units in the integer are always less by one than the places in the corresponding number. This integer is called the *index*, because it points out how many figures are in the number.

TO FIND THE LOGARITHM OF A NUMBER FROM THE TABLES.

In the large tables extending to 100000, the natural numbers from 1000 to 10000 are marked on the margin; but in the common tables only those from 100 to 1000; and in both 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, are marked above the columns.

The logarithms of numbers under 1000 in the large tables, or under 100 in the common, are given in their order at the beginning; and the logarithms of numbers consisting of one place more are found against that number in the column titled 0.

To find in the common tables the logarithm of any number. Look for the three highest figures in the margin on the left hand, and running along that line to the column which has the fourth figure at the top, you will find the logarithm for these four figures. If the number consists of more than four figures, take the difference between the logarithm thus found and the next greater, and multiply it by the remaining figures, and from the product cut off as many figures as are in the multiplier; the rest added to the logarithm for the first four figures gives the logarithm required. The index is not given in the tables, but it is always one less than the number of integers in the given number.

1. Required the logarithm of 73284.

Look in the margin for 732, and on that line in the column which has 8 at the top you will find ·8649855, the logarithm of 7328, and the difference between it and the next logarithm is 592, which, multiplied by 4, gives 2368: therefore, adding 237 to ·8649855, we have 4·8650092 for the logarithm of 73284, with 4 for an index, because the number has five places. If the number had been 732·84, the logarithm would have been the same, but the index would have been 2.

Since the logarithm of 1 is 0, the index of the logarithm of a decimal must be negative. If there be no ciphers after the decimal point, the index is — 1; if there be one cipher, the index is — 2, and so on. A negative index is to be added

when the logarithm is subtracted, and subtracted when the logarithm is added. Sometimes 9 is put for the index of a decimal when there are no ciphers after the decimal point, 8 when there is one cipher, and so on.

2. Required the log. of 6·1953.	.	Ans. 0·7920623.
3. . . . . of 47·5384.	.	1·6770445.
4. . . . . of ·003825.	.	— 3·5826314.

TO FIND THE NUMBER CORRESPONDING TO A GIVEN LOGARITHM.

If the given logarithm be found in the table, the three first figures of the number will be found on the same line in the margin, and the fourth at the top of the column. But if the logarithm be not found exactly in the table, take the number answering the next less, and subtract this logarithm from the given one, and also from the next greater in the table; and, annexing ciphers to the first remainder, divide it by the other, to get the fifth, sixth, &c. figures. The integer places must be one more than the units in the index, and the rest are decimals.

5. Required the number corresponding to the logarithm 4·5971794.

The next less logarithm is ·5971465, and the number answering to it is 3955; the difference between it and the given logarithm is 329, and between it and the next greater in the table is 1098. Divide 3290 by 1098, and the quotient 3, annexed to 3955, gives 39553 for the number sought.

6. Required the number of log. 3·7742395.		Ans. 5946·2.
7. . . . . 2·1475217.	.	140·45.
8. . . . . — 2·8624892.	.	0·07286.

TO FIND THE ARITHMETICAL COMPLEMENT.

Subtract the logarithm from 10, an integer, or subtract the right-hand figure from 10, and all the rest from 9.

9. Thus the arithmetical complement of 3·6427535 is 6·3572465.

10. Required the ar. co. of 2·7493672.		Ans. 7·2506328.
11. . . . . of 1·3607968.	.	8·6392032.

TO PERFORM MULTIPLICATION BY LOGARITHMS.

Add the logarithms of the factors; the sum is the logarithm of the product.

12. Multiply	37·68	log. 1·5761109
by	9·25	log. 0·9661417
Product	348·54	log. 2·5422526

3. Multiply 5·735, 0·023, and 56·25 together.

5·735	log.	0·7585334
0·023	log. —	2·3617278
56·25	log.	1·7501225
Product 7·419655	log.	0·8703837

4. Required the product of 7·542 by ·963. Ans. 7·2629.
5. . . . . ·00352 by ·864. . 0·00304.
6. . . . . ·0925 by 73·5. . 6·7988.

TO PERFORM DIVISION BY LOGARITHMS.

ubtract the logarithm of the divisor from that of the dend: the remainder is the logarithm of the quotient.

r add the arithmetical complement of the divisor to the rithm of the dividend: the sum, with its index lessened 10, is the logarithm of the quotient.

7. Divide 9·7128	log. 0·9873444	log.	0·9873444
by 0·456	log. 9·6589648	ar. co.	0·3410352
Quotient 21·3	log. 1·3283796	log.	1·3283796

8. Required the quotient of 9 by 75. . Ans. ·12.
9. . . . . 8964 by 3·84. . 2334·375.
0. . . . . 62·78 by 71·6. . ·876816.

TO WORK PROPORTION BY LOGARITHMS.

Add the logarithms of the second and third terms together, from their sum subtract the logarithm of the first: the ainder is the logarithm of the fourth term, or answer.

Or add together the arithmetical complement of the first n, and the logarithms of the other two: the sum, with its ex lessened by 10, is the logarithm of the answer.

1. First 36	log. 1·5563025	ar. co.	8·4436975
Second 144	log. 2·1583625	log.	2·1583625
Third 28	log. 1·4471580	log.	1·4471580
	3·6055205		
Fourth 112	log. 2·0492180	log.	2·0492180

2. If 17 men do a piece of work in 28 days, in what time 12 do it? Ans. 39⅔ days.

3. If 13¼ cwt. be carried 57 miles for £2·568, how far uld 34½ cwt. be carried for £8·56? Ans. 72·971 miles.

TO INVOLVE A NUMBER BY LOGARITHMS.

Multiply the logarithm by the name of the power: the duct is the logarithm of the power.

24. Numb. 32 log. 1·5051500 numb. ·009 log. — 3·
3

3d powr. 32768 log. 4·5154500 ·000000729 log. — 7·

NOTE. After multiplying the negative index, the to it from the logarithm must be subtracted from the If the positive index be used, 10 times the name of t lessened by 1 must be taken from the index of the p

25. Number ·0437 log. — 2·6404814, or 8·6404
4

4th power ·00000365 log. — 6·5619256 4·5619

TO EXTRACT THE ROOT OF A NUMBER BY LOGAR

Divide its logarithm by the name of the root: the is the logarithm of the root.

NOTE. If the given number be a decimal, and positive, prefix the name of the root lessened by index, before dividing. If the index be negative, the least number that will make the sum divisibl name of the root: the quotient is the index of the r in dividing the logarithm, the number added only considered as the index.

26. Number ·00130321 log. 4)— 3·1150144 log. 37
1

Fourth root ·19 log. — 1·2787536 log. 9

27. Number 9261 log. 3)3·9666579

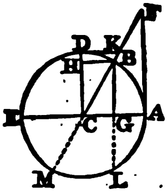
Cube root 21 log. 1·3222193

28. Required the square root of ·5329. .
29. . . . . cube root of ·041063625. .
30. . . . . fourth root of 7. . .

EXERCISES.

1. Reqd. the seventh power of 7·142. Ans
2. . . . sixth root of 2. 1
3. . . . ninth power of ·0375. 12 ciphers be
4. . . . eighth root of ·02405.
5. . . . compound interest of £67·495 for 5½ 4 per cent. Ans.
6. . . rate of comp. int. at which £136·78 5¼ years, amount to £173·564. A
7. . . . time in which £53·5 will amount to at 3½ per cent. comp. int. Ans. 10·

PLANE TRIGONOMETRY.



TRIGONOMETRY is the method of determining the sides and angles of triangles, and of expressing them in known measures. This is done by means of the ratios which certain straight lines in and about the circle have to its radius.

DEFINITIONS.

1. The SINE BG of an arc AB, is a straight line drawn from B, one of its extremities, perpendicular to the diameter AE, which passes through the other.

2. The VERSED SINE AG of an arc AB, is the portion of the diameter AE upon which the sine is perpendicular, between the sine and the arc.

3. The TANGENT AF of an arc AB, touches the circle at A, one of the extremities of the arc, and meets at F the diameter MB, which passes through the other extremity B.

4. The SECANT CF of an arc AB, is a straight line drawn from C the centre, to F the farthest extremity of the tangent.

5. The sine, versed sine, tangent, and secant of an arc AB, are called the sine, versed sine, tangent, and secant of the angle ACB measured by that arc to the radius AC.

6. The SUPPLEMENT of an arc AB, or of an angle ACB, is the difference between it and 180°. Thus BE or AM is the supplement of AB, and BCE or ACM the supplement of ACB.

Cor. 1. An arc or angle, and its supplement, have the same sine, tangent, and secant; for BG is the sine of BE or BCE, AF the tangent of AM or ACM, and CF the secant of AM or ACM.

Cor. 2. The versed sine EG of BCE, together with AG the versed sine of ACB, is equal to the diameter AE.

7. The COMPLEMENT of an arc AB, or angle ACB, is the difference between it and 90°. Thus BD or BCD is the complement of AB or ACB.

8. The sine, versed sine, tangent, and secant of the complement of an arc or angle, are called the cosine, coversed

sine, cotangent, and cosecant of the arc or angle. Thus BH or CG is the cosine of AB or ACB, DH is its coversed sine, DK its cotangent, and CK its cosecant.

Cor. 1. The cosine CG, together with the versed sine AG, is equal to the radius AC.

Cor. 2. The sine BG of an arc AB, is half of BL, the chord of BAL the double of AB.

Cor. 3. The radius is equal to the sine or versed sine of 90°, and to the tangent or cotangent of 45°.

NOTE 1. In what follows, we generally use sin. for sine, cos. for cosine, tan. for tangent, sec. for secant, ver. for versed sine, cov. for coversed sine, cot. for cotangent, cosec. for cosecant, cho. for chord, R. or rad. for radius, and D. or dia. for diameter.

NOTE 2. For the purpose of performing arithmetically the operations of trigonometry, a circle has been selected of which the radius is very large, such as 100000, &c.; and the sines, tangents, &c. have been calculated for every second of the quadrant of such a circle, and arranged in tables; and from these the sines, tangents, &c. for arcs of other circles may be found by proportion.

OF THE TABLES OF SINES, TANGENTS, AND SECANTS.

The common tables have the degrees at the top, and the minutes on the left side, when the degrees are less than 45°; but if greater, the degrees are marked at the bottom, and the minutes on the right side.

The logarithms of the natural sines, tangents, &c. have been taken, and placed in similar tables. These form the tables of artificial sines, tangents, &c. which supply the place of the natural ones in the same way that the logarithms supply the place of natural numbers.

1. Required the artificial sine of $37^\circ\ 23'\ 12''$.

Turn to the page which has 37° at the top, and come down the column titled *Sine* at the top, to the line that has $23'$ on the left side, and you will find 9·7832922, the sine of $37^\circ\ 23'$; and the difference between it and the sine of $37^\circ\ 24'$ is 1653. Then as $60''$ is to $12''$, so is 1653 to 331, the proportional difference for $12''$, which, added to 9·7832922, gives 9·7833253, the sine of $37^\circ\ 23'\ 12''$.

2. Required the degrees and parts of a degree of which 10·2738462 is the artificial tangent.

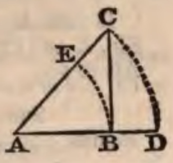
Look for the nearest tangent 10·2737163, and because it is titled *Tang.* at the bottom, take the degrees at the foot, and the minutes on the right side, where are found $61^\circ\ 58'$. The difference between this tangent and the one above it is 3046,

and the difference between it and the given one is 1299; therefore 3046 : 1299 : : 60″ : 26″, so that 10·2738462 is the tangent of 61° 58′ 26″.

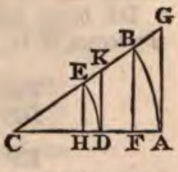
3. Natural sine of 57° 26′ 20″.	.	Ans. ·8428179.
4. Artificial cosine of 67° 31′ 40″.	.	9·5823310.
5. Artificial secant of 73° 27′ 45″.	.	10·5456998.
6. Natural cosine is ·7476822.	. .	41° 36′ 36″.
7. Artificial secant is 10·475546.	.	70° 28′ 20″.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

The first thing to be done in resolving right-angled triangles is to make one of the sides the radius of a circle, the centre of which is at an acute angle, and thus to determine what the other sides would be in that circle.



If from the centre A, with the radius AC, the arc CD be described, then BC will be the sine of CAB, and AB its cosine. But if the centre be at C, and the circle pass through A, then AB is the sine of C, and BC its cosine. Hence when the hypotenuse is radius, the other sides are the sines of their opposite angles, or the cosines of their adjacent angles. Again, if from the centre A, with the radius AB, the arc BE be described, then BC is the tangent of A, and AC is its secant.

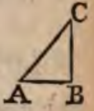


Suppose ACB any angle, and AB an arc described with the radius of the circle, from which the sines, tangents, &c. in the tables were calculated; then BF is the sine in the tables, CF the cosine, AG the tangent, and CG the secant in the tables. Let CEH be a right-angled triangle. If CE be radius, EH will be the sine of C, and CH its cosine. But the triangles CEH, CBF, being similar, CE : EH : : CB : BF; that is, as CE is to EH, so is the radius of the tables to the sine of C in the tables. In like manner CE is to CH as the radius to the cosine in the tables. In the same way it may be shown, that if CDK were the triangle, and CD the radius, CD is to DK as the radius to the tangent of C in the tables, and that DC is to CK as the radius is to the secant of C in the tables; so that after determining the names of the sides of the triangle, any two sides are to one another as their names in the tables.

The terms of the proportion, however, must be so arranged, that the thing required shall be the last term, thus:

To find EH,	R : sin. C :: CE : EH
To find CE,	sin. C : R :: HE : EC
To find C,	CE : EH :: R : sin. C.

And these three are all the variations which are requisite. But the student should accustom himself to state them without hesitation.



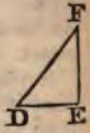
1. In the triangle ABC, right-angled at B, are given the hypotenuse AC 324 feet, and the angle BAC 48° 17′; to find the base AB, and perpendicular BC.

NOTE. When one of the acute angles is known, the other is got by subtracting that one from 90°.

If AC be radius, and A the centre, CB is the sine of A, and AB its cosine. Wherefore,

R : sin. A :: AC : CB, and R : cos. A :: CA : AB.

Sin. A 48° 17′	log. 9·8729976	cos. A	log. 9·8231138
AC 324	log. 2·5105450		2·5105450
Sum	12·3835426		12·3336588
Radius	log. 10·0000000		10·0000000
CB 241·85	log. 2·3835426	AB 215·6	log. 2·3336588



2. Given DE 1254 feet, and the angle D 51° 19′; to find the hypotenuse DF, and the perpendicular EF.

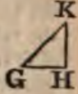
DE being radius, EF is the tangent and DF the secant of D.

R : tan. D :: DE : EF.

Tan. D 51° 19′ — R.	log. 0·0965445
DE 1254	log. 3·0982975
EF 1566·18	log. 3·1948420

R : sec. D :: ED : DF.

Sec. D 51° 19′ — R.	log. 0·2041091
DE 1254	log. 3·0982975
DF 2006·35	log. 3·3024066



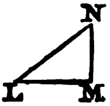
3. Given the angle G 43° 38′, and the opposite side HK 186 feet; to find the hypotenuse GK, and the base GH.

This may be wrought as the last, by first finding GKH. Or, GK being radius, KH is the sin. G; and GH being radius, HK is tan. G.

Sin. G : R :: HK : KG, and tan. G : R :: KH : HG.

HK 186+R.	log. 12·2695129	HK+R.	log. 12·2695129
Sin. G 43° 38′	log. 9·8388747	tan. G	log. 9·9792738
GK 269·549	log. 2·4306382	GH 195·09	log. 2·2902391

4. Given the hypotenuse LN 415 inches, and the perpendicular MN 249; to find the angles, and LM.



LN : NM :: R : sin. L.

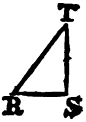
NM 249+R.	log. 12·3961993
LN 415	log. 2·6180481
Sin. L 36° 52′ 12″	log. 9·7781512

R : cos. L :: NL : LM.

Cos. L 36° 52′ 12″ — R.	log. — 1·9030894
LN 415	log. 2·6180481
LM 332	log. 2·5211375

NOTE. LM is equal to the square root of the product of the sum and difference of LN and NM $= \sqrt{664 \times 166} = \sqrt{110224} = 332$.

5. Given the base RS 53 miles, and the perpendicular ST 67; to find the angles, and hypotenuse RT.



RS : ST :: R : tan. R.

ST 67+R.	log. 11·8260748
RS 53	log. 1·7242759
Tan. R 51° 39′ 16″	log. 10·1017989

R : sec. R :: SR : RT.

Sec. R 51° 39′ 16″ — Rad.	log. 0·2073261
RS 53	log. 1·7242759
RT 85·4284	log. 1·9316020

NOTE. The square of RT is equal to the sum of the squares of RS and ST; therefore TR $= \sqrt{53^2 + 67^2} = \sqrt{7298} = 85{\cdot}4284$.

6. Given the hypotenuse 893, and the base 586 chains.
Ans. Angle at base 48° 59′ 17″, perpendicular 673·832 ch.

7. Given the base 326 yards, and the vertical angle 64° 40′.
Ans. Hypotenuse 360·686, perpendicular 154·33 yards.

8. Given the perpendicular 286, and vertical angle 71° 24′.
Ans. Hypotenuse 896·666, base 849·832.

9. Given the hypotenuse 963 links, and vertical angle 41° 48′. Ans. Base 641·87, perpendicular 717·89 links.

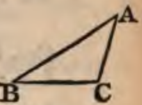
OBLIQUE TRIANGLES.

If two angles of a triangle be known, the third is got by subtracting their sum from 180° ; and if one angle be known, the sum of the other two is got by subtracting it from 180°.

Rule I. Any two sides of a triangle are to one another as the sines of the angles opposite to them. Thus BC : CA :: sin. A : sin. B, or sin. A : sin. B :: CB : CA. The former order is to be used when an angle is required, and the latter when a side.

Note. This rule is to be used whenever a given angle is opposite to a given side.

1. Given two sides AB 532, and BC 358 feet, and the angle at C 107° 40′; to find the angles at A and B, and AC. The figure is drawn by Prob. XX. Practical Geometry.



AB : BC : : sin. C : sin. A.

Sin. C (107° 40′) 72° 20′*	log. 9·9790192
BC 358 feet	log. 2·5538830
	12·5329022
BA 532	log. 2·7259116
Sin. A 39° 53′	log. 9·8069906

B = 180° — (C + A), and sin. C : sin. B : : BA : AC.

Sin. B 32° 27′	log. 9·7296211
BA 532	log. 2·7259116
	12·4555327
Sin. C 107° 40′	log. 9·9790192
AC 299·6	log. 2·4765135

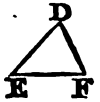
2. Given AB 232, and BC 345 yards, and the angle at C 37° 20′.

By proceeding in the same way, the angle at A may be either 64° 24′ or 115° 36′, and therefore the angle at B may be either 78° 16′ or 27° 4′, and AC 374·56 or 174·07. For AB being less than BC, there are two triangles which have each of them the given things in them.

3. Two places are 560 feet from one another, and at a station 258 feet from the first place, their distance subtended an angle of 63° 28′. Required the distance of the station from the other. Ans. 625·469 feet.

* When the angle is greater than 90°, take the sine, tangent, &c. of its supplement.

4. Given two angles D 63° 48′, and E 49° 25′, and the side EF opposite to D 275 yards; to find DE and DF. Constructed by Prob. XXI. Practical Geometry. The angle at F is $= 180° - (D+E) = 66° 47′$.



Sin. D : sin. E :: EF : FD.		
Sin. E 49° 25′	log.	9·8805052
EF 275	log.	2·4393327
		12·3198379
Sin. D 63° 48′	log.	9·9529175
FD 232·77	log.	2·3669204
Sin. D : sin. F :: FE : ED.		
Sin. F 66° 47′	log.	9·9633253
EF 275	log.	2·4393327
		12·4026580
Sin. D 63° 48′	log.	9·9529175
DE 281·67	log.	2·4497405

5. Given the angles at E 49° 25′, and F 63° 48′, and the side EF 275; to find ED and DF.

Ans. ED 268·488, and DF 227·2546.

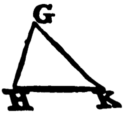
6. A ship sailing due north observes a cape bearing N. 54° 12′ W.; and after sailing 27 miles, the cape bore S. 70° 30′ W. Required her distances from it.

Ans. First distance 30·957, second distance 26·636 miles.

Rule II. When two sides and the angle between them are given.

Add and subtract the sides to get their sum and difference. Subtract the angle from 180°, and take half the remainder, to get half the sum of the unknown angles. Then as the sum of the sides is to their difference, so is the tangent of half the sum of the unknown angles to the tangent of half their difference. Having thus found the half difference, add it to the half sum to get the angle opposite to the greater side, and subtract it to get the less angle; after which the third side is found by Rule I.

7. Given the sides GH 133, and HK 176 yards, and the angle at H 73° 16′; to find the angles at G and K, and the side GK.



KH + HG : KH — HG : : tan. $\frac{1}{2}$(G + K) : tan. $\frac{1}{2}$(G — K).

KH — HG 43	log.	1·6334685
Tan. $\frac{1}{2}$(180° — H) 53° 22′	log.	10·1286790
		11·7621475
KH + GH 309	log.	2·4899585
Tan. $\frac{1}{2}$(G — K) 10° 36′	log.	9·2721890
Angle G 63° 58′		
Angle K 42° 46′		

Sin. G : sin. H : : HK : KG.

Sin. H 73° 16′	log.	9·9812091
HK 176	log.	2·2455127
		12·2267218
Sin. G 63° 58′	log.	9·9535369
GK 187·58	log.	2·2731849

8. Given GH 237, and GK 482 feet, and the angle at G 77° 48′; to find the angles at H and K, and HK.

Ans. H 73° 59′ 39″, K 28° 12′ 21″, and HK 490·1144 feet.

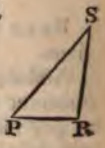
9. Given HK 78, and KG 168, and the angle K 128° 26′.

Ans. H 35° 48′ 20″, G 15° 45′ 40″, HG 224·94.

Rule III. When the three sides are given.

Add the three sides, and from half the sum subtract the side opposite to the angle sought; then take the arithmetical complements of the two sides containing the angle sought, and the logarithms of the half sum and of the remainder, and add these four together, and half the sum will be the cosine of half the angle sought.

10. Given the sides SP 230, PR 365, and SR 426 feet; to find the angles.



SP 230	ar. co.	7·6382722
PR 365	ar. co.	7·4377071
SR 426		
$\frac{1}{2}$)1021		
$\frac{1}{2}$ Sum 510·5	log.	2·7079957
426		
Rem. 84·5	log.	1·9268567
		$\frac{1}{2}$)19·7108317
$\frac{1}{2}$P 44° 12′ 24″	cosine	9·8554158
P 88° 24′ 48″		

In the same manner the angle S is 58° 55′ 25″.

11. Given the sides SP 1248, PR 728, and RS 956 feet.
Ans. The angle R 94° 40′ 50″, P 49° 46′ 16″.

12. Given SP 375, PR 275, and RS 196.
Ans. The angle S 45° 17′ 26½″, P 30° 25′ 57½″.

PROMISCUOUS EXAMPLES.

1. Given the hypotenuse of a right-angled triangle 528 feet, and one of the acute angles 39° 27′.
Ans. The opposite side 335·57, adjacent side 407·7 feet.

2. Given the base 256, and the adjacent angle 57° 28′.
Ans. Hypotenuse 476·022, perpendicular 401·324 feet.

3. Given the perpendicular 297 feet, and the angle at the base 36° 48′. Ans. Hypotenuse 495·8, base 397 feet.

4. Given the hypotenuse 1268, and perpendicular 428 yards.
Ans. The base 1193·583, adjacent angle 19° 43′ 37·3″.

5. Given the base 674, and the perpendicular 438 yards.
Ans. Hypotenuse 803·816 yards, angle at base 33° 1′ 4″.

6. Given the hypotenuse 97, and the base 38 miles.
Ans. Perpendicular 89·247 miles, angle at base 66° 56′ 11″.

7. Given the base 326, and the vertical angle 67° 30′.
Ans. The hypotenuse 352·86, perpendicular 135·034.

8. In an oblique triangle, given two angles 46° 48′ and 114° 26′, and the side opposite the lesser 254 feet.
Ans. Other sides 317·233 and 112·097 feet.

9. Given two angles 56° 24′ and 74° 28′, and the side between them 354. Ans. Other sides 451·011 and 389·898.

10. Given two sides 572 and 748, and the angle opposite to the greater 67° 30′.
Ans. Angle opposite less 44° 57′ 2″, third side 748·267.

11. Given two sides 356 and 294, and the angle opposite to the lesser 51° 27′.
Ans. Other angles 71° 15′ 38·2″ and 57° 17′ 21·8″, or 108° 44′ 21·8″ and 19° 48′ 38·2″; third side 316·31 or 127·407.

12. Given two sides 1864 and 1235, and included angle 73° 38′.
Ans. Other angles 68° 21′ 15·4″ and 38° 0′ 44·6″, third side 1924·2.

13. Given two sides 436 and 219, and included angle 127°.
Ans. Other angles 35° 52′ 45·7″ and 17° 7′ 14·3″, third side 594·15.

14. Given the three sides 456, 327, and 184 yards.
Ans. Angles 123° 55′ 10·6″, 36° 31′ 3·2″, and 19° 33′ 46·2″.

15. Given the sides 2586, 1482, and 1234.
Ans. Angles 144° 14′ 53″, 19° 33′ 47″, and 16° 11′ 20″.

MENSURATION OF SUPERFICIES.

THE Imperial Yard is the distance between the centres of the points in the gold studs fixed in the brass rod belonging to the House of Commons, and titled, "Standard Yard, 1760." When used, the brass must be at the temperature of 62 degrees of Fahrenheit's thermometer.

This yard is divided into 3 feet, and each foot into 12 inches; $5\frac{1}{2}$ yards make a pole, 40 poles make a furlong, and 8 furlongs a mile.

The length of a pendulum vibrating seconds of mean time, at the level of the sea, in the latitude of London, contains 39·1393 such inches.

A square described upon a straight line, of which the length is an inch, is called a *square inch;* and the same is to be understood of a square foot, &c.

The *area* of a surface is the number of square inches, feet, &c. which it contains.

Land is estimated by the acre. In England, 640 acres make a square mile; and the acre is subdivided into 4 roods, each 40 perches or square poles; and the perch consists of $30\frac{1}{4}$ square yards. A square yard is 9 square feet, and a square foot is 144 square inches. The acre contains 10 square chains, each 16 perches or 100000 square links. The length of the English chain is 66 feet, and it is divided into 100 links, each 7·92 inches.

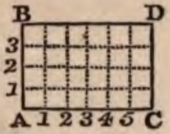
The Scotch acre is also divided into 4 roods, each of them 40 falls; and a fall contains 36 square Scotch ells, and a square ell 1369 square inches, = 1373·392 English inches. The Scotch ell contains 37 Scotch inches, or 37·0593 imperial inches. The Scotch chain is 74·1196 imperial feet; and consequently a Scotch acre is equal to 1·26118345 imperial acre.

PARALLELOGRAMS.

PROB. I. To measure a right-angled parallelogram.

RULE. Multiply one of the sides by the other.

1. Required the area of the rectangle ABDC, of which the sides are AB 4 yards, and AC 6.



$$\begin{array}{r} 6 \\ 4 \\ \hline \end{array}$$

Area 24 square yards.

If AC be divided into 6 equal parts or yards, and AB into 4, and lines be drawn parallel to the sides, the rectangle will be divided into 24 squares, each of them a yard.

2. Required the area of a square, each side 37 feet.

Ans. 1369 square feet.

3. Required the area of a rectangle, the sides 326 and 153 feet.

$$\begin{array}{r} 326 \\ 153 \\ \hline 9\,|\,49878 \text{ square feet.} \\ \hline 30\tfrac{1}{4})\ 5542 \text{ square yards.} \\ \hline 40\,|\,184—6\tfrac{1}{4} \\ \hline 4\,|\,4—23 \\ \hline \end{array}$$

Ans. 1 acre 0 roods 23 perches $6\frac{1}{4}$ yards.

4. Required the area of a square, each side 3525 links.

Ans. 124 ac. 1 ro. 1 per.

5. Required the area of a square, the diagonal being 56 Scotch ells.

Multiply the diagonal by its half.

Ans. 1568 Sc. ells, = 1 rood 3 falls 20 ells.

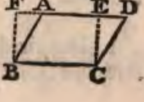
6. A rectangular space, 68 feet 3 inches long by 56 feet 8 inches broad, is to be paved with stones each 2 feet 3 inches by 10 inches. Required how many stones it will take, and what will be the expense at 2s. 3d. for a square yard.

Ans. $2062\frac{2}{3}$ stones, expense £48, 6s. $10\frac{1}{2}$d.

Prob. II. To measure any parallelogram.

Rule. Multiply one of the sides by the perpendicular dropt upon it from the opposite side. See Appendix, Prop. 14, Schol.

1. Required the area of the parallelogram ABCD, of which the sides are AB 214, and BC 354, and the perpendicular CE 192 feet.



$$\begin{array}{r} 354 \\ 192 \\ \hline 9\,|\,67968 \text{ square feet.} \\ \hline 4840)\,7552 \text{ square yards.} \\ \hline \end{array}$$

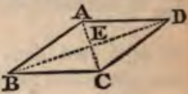
Ans. 1 acre 2 roods 9 perches $19\frac{3}{4}$ yards.

The triangle ABF = DCE; therefore ABCD = rectangle CEFB.

2. Required the area of a rhombus, the side 358, and the perpendicular on it 194 feet.

Ans. 69452 feet, = 1 ac. 2 ro. 15 per. 3 yds. 5 feet.

3. Required the area of a rhombus, of which the diagonals are AC 436, and BD 623 yards.



Note. AC and BD bisect one another at right angles.

Ans. $AE \times BD = 623 \times 218 = 135814$ yards, = 28 ac. 9 per. $21\frac{5}{4}$ yds.

4. Required the area of a rhomboid, the sides 1234 and 762, and the perpendicular on the former 658 links.

Ans. 8 acres 19 perches 4 yards $6\frac{1}{4}$ feet.

5. Required the area of a parallelogram, the sides 56 feet 8 inches and 42 feet 10 inches, and the perpendicular on the latter 47 feet 3 inches. Ans. 2023 feet $10\frac{1}{2}$ inches.

6. Required the area of a rhomboid, the sides 24 and 18 poles, and the perpendicular upon the latter 96 yards.

Ans. 1 acre 3 roods 34 poles $5\frac{1}{2}$ yards.

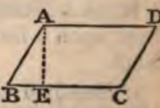
7. Required the area of a rhombus, the diagonals $6\frac{1}{2}$ feet and $3\frac{1}{4}$ feet. Ans. 10 feet 81 inches.

Prob. III. Given two sides and an angle of a parallelogram; to find the area.

Rule. Multiply the product of the two sides by the natural sine of the angle. See Appendix, Prop. 14, Schol.

Or add the logarithms of the sides and the logarithm sine of the angle: the sum, after taking 10 from the index, will be the logarithm of the area.

For $AB \times \sin. B =$ perpendicular AE; therefore $AB \times \sin. B \times BC = AE \times BC$ the area.



1. Required the area of the rhomboid ABCD, of which the sides are AB 278, and BC 456 feet, and the angle B 58° 46′.

$$
\begin{array}{r}
\text{Sin. } 58^\circ\ 46' = \cdot 85506 \\
456 \\
\hline
389\cdot 90736 \\
278 \\
\hline
43560)\ 108394\cdot 24608 \text{ square feet.}
\end{array}
$$

Ans. 2 acres 1 rood 38 perches $4\frac{1}{8}$ yards.

2. Required the area of a rhombus, the side 172 ells, and an angle 72° 30′.

Ans. $2\cdot 235528 \times 2 + 9\cdot 979420 = 4\cdot 450476$ log. of 28215 ells.

3. Required the area of a rhomboid, the sides 136 and 97 yards, and the angle 73° 16′.

Ans. 2 acres 2 roods 17 perches 19 yards $1\frac{1}{4}$ feet.

4. Required the area of a rhomboid, the sides 628 and 425 links, and the angle 126°.

Ans. 2 acres 25 perches 14 yards 5·4 feet.

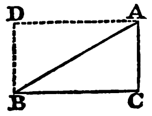
5. Required the area of a rhombus, the side 57 poles, and the angle 67° 45′. Ans. 18 ac. 3 ro. 7 per. 2·4 yds.

6. Required the area of a rhombus, the side 157 inches, and the angle 29° 12′. Ans. 83 feet $73\frac{1}{4}$ inches.

TRIANGLES.

Prob. IV. Given the base and the perpendicular of a triangle; to find the area.

Rule. Multiply the base and perpendicular, the one by half of the other.

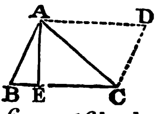


For a triangle ABC is half a parallelogram BCAD, which has the same base and perpendicular. See Appendix, Prop. 14, Schol.

1. Required the area of the right-angled triangle ABC, of which the sides about the right angle are BC 254, and AC 136 yards.

$$
\begin{array}{r}
254 \\
68 \\
\hline
4840)\ 17272 \text{ square yards.}
\end{array}
$$

Ans. 3 acres 2 roods 10 perches $29\frac{1}{2}$ yards.



2. Required the area of a triangle ABC, the base CB 396, and side AB 278, and perpendicular AE 174 feet.

Ans. $396 \times 87 = 34452$ square feet, $=$ 3 ro. 6 per. $16\frac{1}{2}$ yds.

K

3. Required the area of a triangle, one angle 43°, adjacent side 296, and perpendicular on it 176 yards.

Ans. 26048 yards, = 5 ac. 1 ro. 21 per. $2\frac{3}{4}$ yds.

4. Required the area of a triangle, the sides 156 and 97 poles, and the perpendicular upon the latter 102 poles.

Ans. 30 acres 3 roods 27 perches.

5. Required the area of a triangle, the side 684 links, the angle adjacent 137°, and the perpendicular 928 links.

Ans. 3 acres 27 perches $24\frac{1}{4}$ yards.

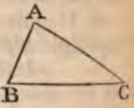
Prob. V. Given two sides and the included angle of a triangle; to find the area.

Rule. Multiply one side by half of the other, and by the natural sine of the included angle. See Appendix, Prop. 14, Schol.

Or add the logarithms of one side and of half the other, and the logarithm sine of the angle: the sum, rejecting 10 in the index, is the logarithm of the area.

This rule is evident from Prob. III.

1. Required the area of the triangle ABC, of which AB is 534, and BC 872 links, and the angle B 63° 40′.



$$\begin{array}{r} \text{Sin. } 63^\circ\ 40' = \cdot 89623 \\ 872 \\ \hline 781\cdot 51256 \\ 267 \\ \hline 100000)\ 208663\cdot 85352 \text{ square links.} \\ \hline 2\cdot 0866385 \\ \hline \end{array}$$

Ans. 2 acres 13 perches 26 yards.

2. Required the area of a triangle having an angle 78° 30′, and the containing sides 933 and 471 Scotch links given.

Ans. 215310·5 links, = 2 acres 24 falls 17·88 ells.

3. Required the area of a triangle, two sides 12 feet 9 inches, and 7 feet 3 inches, included angle 57° 38′.

Ans. 5621·5 inches, = 4 yards 3 feet $5\frac{1}{2}$ inches.

4. Required the area of a triangle, an angle 54° 30′, and the containing sides 328 and 157 yards.

Ans. 4 acres 1 rood 12 perches 29 yards.

5. Required the area of a triangle, an angle 128°, and the sides about it 38 and 93 poles. Ans. 8 ac. 2 ro. 32 per. $12\frac{1}{2}$ yds.

6. Required the area of a triangle, an angle 17° 54′, and the adjacent sides 27 and 12 miles. Ans. 49·79177 miles.

7. Required the area of a triangle, an angle 93°, and the sides about it 137 and 428 ells. Ans. 5 ac. 13 falls 9·82 ells.

PROB. VI. Given the three sides of a triangle; to find the area.

RULE. Add the three sides together, and from half the sum subtract each side separately. Then multiply the half sum and the three remainders successively, and the square root of the last product will be the area.

Or add the logarithms of the half sum and of the three remainders, and half the sum will be the logarithm of the area. See Appendix, Prop. 41, Cor.

1. Required the area of the triangle ABC, of which the sides are AB 221, BC 255, and AC 238 feet.

$$(255+221+238)\times\tfrac{1}{2}=357$$
$$357-255=102$$
$$36414$$
$$357-221=136$$
$$4952304$$
$$357-238=119$$

Ans. 5,89,32,41,76 (24276 square feet, = [2 ro. 9 per. $5\frac{1}{12}$ yds.

44) 189

482) 1332

4847) 36841

48546) 291276

2. Required the area of a triangle, the sides 834, 658, and 423 links.

The half sum	957·5	log. 2·9811388
First rem.	123·5	log. 2·0916670
Second rem.	299·5	log. 2·4763968
Third rem.	534·5	log. 2·7279477
		2)10·2771503

Area 137586·3 links log. 5·1385752
= 1 acre 1 rood 20 perches 4 yards 1·6 feet.

3. Required the area of an isosceles triangle, the equal sides 156, and the third side 78 yards.

Ans. $39\sqrt{(156+39)(156-39)} = 39\sqrt{195\times 117} =$ 5890·8 yards area, = 1 acre 34 perches 22 yards 2·7 feet.

4. Required the area of an equilateral triangle, each side 34 inches. Ans. $17\times 17\times\sqrt{3} = 500{\cdot}56268$ square inches area.

5. Required the area of a triangle, the sides 56, 52, and 60 yards. Ans. 1344 yards.

6. Required the area of a parallelogram, the sides 432 and 263, and a diagonal 342 feet.

Ans. 89945·66 square feet, = 2 acres 10 perch. 11·46 yards.

7. Required the area of a triangle, one side 956 links, and each of the other two 627 links.

Ans. 1 acre 3 roods 30 perches 10 yards.

8. Required the area of a rhomboid, the sides 57 and 83 poles, and the diagonal 127 poles.

Ans. 22 acres 3 roods 21 perches 26 yards 5 feet.

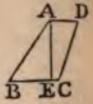
QUADRILATERALS.

Prob. VII. To find the area of a trapeziod.

Rule. Multiply half the sum of the parallel sides by the perpendicular from the one to the other.

For the triangles into which it may be divided have the same perpendicular.

1. Required the area of the trapeziod ABCD, of which the parallel sides are AD 96 and BC 143, a third side AB 126 yards, and the perpendicular AE 89 yards.



$$143+96=239$$
$$44\tfrac{1}{2}$$
$$10635{\cdot}5 \text{ yards.}$$

Ans. 2 acres 31 perches $17\frac{3}{4}$ yards.

2. Required the area of a trapeziod, the parallels 786 and 473, another side 1230, and the perpendicular distance 986 links. Ans. 6 acres 33 perches 3 yards.

3. Required the area of a trapeziod, the parallels 564 and 348, a third side 452, and the perpendicular 397 feet.

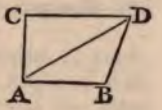
Ans. 4 acres 24 perches $28\frac{2}{3}$ yards.

4. Required the area of a trapeziod, the parallels 93 and 157 poles, angle at the latter 62°, and the perpendicular on it 86 poles. Ans. 67 acres 30 perches.

5. Required the area of a trapeziod, the parallel sides 386 and 294 feet, an angle at the first 43°, and the perpendicular upon the latter 328 feet. Ans. 2 ac. 2 ro. 9 per. 18 yds. $7\frac{3}{4}$ ft.

Prob. VIII. To find the area of any quadrilateral.

Rule. Divide it into triangles, by drawing a diagonal. Find the areas of the triangles separately, and add them: the sum is the area of the figure.



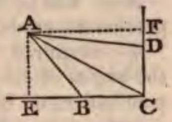
1. Required the area of the quadrilateral ABCD, of which the sides are AC 236, BD 348, AB 392, and DC 427 feet, and the diagonal AD 473.

$$\sqrt{(606{\cdot}5)\times(606{\cdot}5-348)\times(606{\cdot}5-392)\times(606{\cdot}5-473)}=67003{\cdot}90 \text{ DAC}$$

$$\sqrt{(568)\times(568-236)\times(568-427)\times(568-473)}=50259{\cdot}08 \text{ ABD}$$

117262·98 square feet.

Ans. 2 acres 2 roods 30 perches 21 yards $6\frac{1}{2}$ feet.



2. Required the area of the trapeze ABCD, the sides AB 218, BC 194, CD 166 yards, and the perpendiculars from A upon BC 136, and upon CD 152 yards.

Ans. 25808 yards, = 5 acres 1 rood 13 perches $4\frac{5}{4}$ yards.

3. Required the area of a trapeze ABCD, the sides AB 842, BC 938, CD 753, AD 826 links, and the angle A 78° 28′.

By trigonometry BD = 1055·05.

Ans. Area 683885 square links, = 6 ac. 3 ro. 14 per. $6\frac{1}{2}$ yds.

4. Required the area of a trapeze ABCD, three sides AB 543, BC 428, CD 634 links, and the angles B 74° 40′ and C 84° 20′.

By trigonometry BD = 729·077.

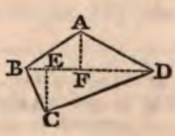
Ans. Area 185392·38 links, = 1 ac. 3 ro. 16 per. 19 yds.

5. Required the area of a trapeze, the four sides 328, 456, 572, and 298, and the diagonal from the angle between the first and second 598 feet. Ans. 3 ac. 1 ro. 31 per. 29 yds. 3·8 ft.

6. Required the area of a trapeze, the diagonal 1268 links, the perpendiculars from one of its extremities upon the opposite sides 784 and 672, and the length of these sides 856 and 548 links. Ans. 5 acres 31 perches 14 yards 6·858 feet.

Prob. IX. Given a diagonal of a quadrilateral, and the perpendiculars upon it from the opposite angles; to find the area.

Rule. Add the perpendiculars together, and multiply half the sum by the diagonal.



1. Required the area of the quadrilateral ABCD, of which the sides are AB 68 and BC 54 yards, the diagonal BD 133, and the perpendiculars AF 37 and CE 44 yards.

$$37 + 44 = 81$$
$$\tfrac{1}{2} \text{ of } 133 = 66\tfrac{1}{2}$$

5386·5 square yards.

Ans. 1 acre 18 perches 2 yards.

2. Required the area of the trapeze ABCD, the sides AB 672, BC 834, the diagonal BD 1296, and the perpendiculars AE 418 and CF 550 links. Ans. 6 ac. 1 ro. 3 per. $18\frac{4}{5}$ yds.

3. Required the area of a parallelogram, of which one of the diagonals is 486 feet, and each of the perpendiculars upon it from the opposite angle 126.

Ans. $486 \times 126 = 61236$ feet area, $=$ 1 acre 1 rood 24 perches 28 yards.

4. Required the area of a trapeze, the diagonal 1356, the angles at one of its extremities 57° and 42°, and the perpendiculars on it 568 and 724 links.

Ans. 8 acres 3 roods 21 perches 2 yards 6 feet.

5. Required the area of a quadrilateral, of which the diagonals cut one another at right angles, the segments of the one are 328 and 523 feet, and of the other 498 and 672.

Ans. 11 acres 1 rood 28 perches 18 yards.

Prob. X. Given the diagonals of a quadrilateral, and the angle at their intersection; to find the area.

Rule. Multiply half the product of the diagonals by the natural sine of the angle.

Or add the logarithms of one diagonal, half the other, and the log. sine of the angle: the sum, lessened by 10 in the index, will be the logarithm of the area.

Note 1. If the angle made by the diagonals be a right angle, half the product of the diagonals is the area.

The triangle $ACD = AED + DEC = \frac{1}{2}AE \times ED \times \sin. E + \frac{1}{2}EC \times ED \times \sin. E = \frac{1}{2}AC \times ED \times \sin. E$; and $ABC = \frac{1}{2}AC \times EB \times \sin. E$.

1. Required the area of the quadrilateral ABCD, of which the diagonals are AC 674 and BD 398 feet, and the acute angle at E 67° 30′.

Nat. sine of $67° 30' = \cdot 92388$
674
622·69512
199

Ans. Area 123916·32888 square feet, $=$ 2 acres 3 roods 15 perches $4\frac{5}{4}$ yards.

2. Required the area of a parallelogram, the diagonals 436 and 324 yards, and their angle 48° 38′.

Ans. 53009 yards, = 10 acres 3 roods 32 perches 11 yards.

3. Required the area of a trapeze, the sides 856 and 643, the diagonal joining their extremities 1154, and the other 1345 links, and the angle made by the diagonals 57° 30′.

Ans. 6 acres 2 roods 7 perches $7\frac{1}{3}$ yards.

4. Required the area of a quadrilateral, the diagonals 72 and 48 feet, and containing a right angle. Ans. 192 yards.

5. The diagonals of a quadrilateral are 567 and 743 links, and they contain an angle of 73° 30′; the side joining their extremities opposite to this angle 324.

Ans. 2 roods 3 perches 4 yards $3\frac{3}{4}$ feet.

6. Required the area of a quadrilateral, the diagonals 924 links and 1256, and they bisect one another in an angle of 52° 30′. Area 4 acres 2 roods 16 perches 17 yards $4\frac{1}{7}$ feet.

Note 2. If the sides be given instead of the diagonals,

Add the squares of each pair of opposite sides, and subtract the less sum from the greater: one-fourth of the remainder, multiplied by the natural tangent of the angle contained by the diagonals, will be the area.* See Appendix, Prop. 42.

Note 3. When the quadrilateral is in a circle, or its opposite angles are together 180°,

From half the perimeter subtract each side separately; multiply the four remainders successively, and the square root of the product will be the area. See Appendix, Prop. 44.

7. Required the area of a quadrilateral, of which the sides are 7, 8, 9, and 10 yards, and the angle contained by the diagonals 80°.

$$10^2+8^2=164$$
$$9^2+7^2=130$$
$$4\,|\,34$$
$$8{\cdot}5$$
$$\text{Nat. tan. } 80^\circ = 5{\cdot}67128$$

Ans. 48·20588 square yards.

8. Required the area of a trapeze in a circle, the sides 326, 438, 247, and 392 feet.

Ans. 117976 square feet, = 2 ac. 2 ro. 33 per. $10\frac{1}{6}$ yds.

---

* If a table of natural tangents be not at hand, multiply by the natural sine, and divide by the natural cosine. Or add the log. of half the remainder to the log. *tangent*: the sum is the log. of the area.

9. Required the area of a quadrilateral in a circle, the sides 24, 26, 28, 30 yards.

Ans. 723·98895 yards, = 23 perches $28\frac{1}{4}$ yards.

10. Required the area of a quadrilateral, of which the opposite angles are together 180°, the sides 40, 55, 60, 75 chains.

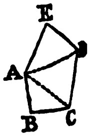
Ans. 3146·427 ch. = 314 ac. 2 ro. 22 per. 25 yds. 1·532 ft.

POLYGONS.

Prob. XI. To find the area of any rectilineal figure.

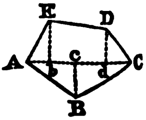
Rule. Draw diagonals so as to divide the figure into quadrilaterals and triangles, and find the areas of these figures separately, and add them: the sum is the area of the whole.

1. Required the area of the pentagon ABCDE, of which the sides are AB 354, BC 432, CD 518, DE 465, and EA 397 feet; and the diagonals AC 574, and AD 612 feet.



By Prob. VI. the triangle . .	ABC	is	76338·2 feet.
	ACD		137791·1
	ADE		92302·3
Whole figure, 7 ac. 5 per. 16 yds. 6·35 feet, =			306431·6

2. In order to obtain the area of the field ABCDE, I measured along the diagonal AC; and at b, 326 links from A, I took the perpendicular bE, 97 links: then I measured to c, 543 links from A, where I took the offset cB 354 links; and measuring on to d, 749 links from A, I took the offset dD 158 links. The whole diagonal AC is 987 links. Required the area.



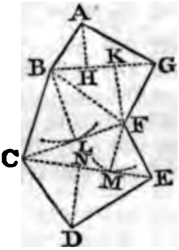
By Prob. VII. $\text{E}bd\text{D} = \frac{1}{2}(\text{E}b + \text{D}d) \times bd =$ 53932·5 links.

By Prob. IV. . . $\text{A}b\text{E} = \frac{1}{2}\text{A}b \times \text{E}b =$ 15811·0

$\text{D}d\text{C} = \frac{1}{2}dc \times d\text{D} =$ 18802·0

$\text{ABC} = \frac{1}{2}\text{AC} \times \text{B}c =$ 174699·0

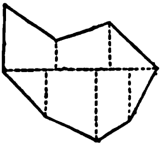
Area of whole, 2 ac. 2 ro. 21 per. 5·78 yds. = 263244·5



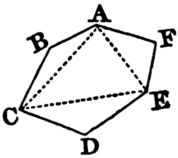
ired the area of the field , of which are given the sides CD 927 links, the diagonals 3F 1037, CF 1284, and CE :he perpendiculars upon BG 7 and FK 384, upon CF is l upon CE are FM 678 and ks.

'. . . BFC $= \frac{1}{2}$CF $\times$ LB $=$ 359520·0 links.
. ABFG $= \frac{1}{2}$(AH+FK)$\times$BG $=$ 479053·5
'CDE $= \frac{1}{2}$(FM+DN) $\times$ CE $=$ 848815·0

:6 ac. 3 ro. 19 per. 24·85 yds. $=$ 1687388·5

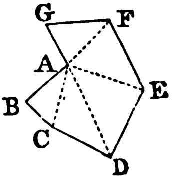


red along a diagonal from east 30 from its east extremity, a .r to it on the south side, of :ached to an angle, and at 380 me extremity a perpendicular side, of 428, reached an angle. :rpendicular of 560 reached an south side; at 812, a perpen- 30 reached an angle on the north; at 1140, a .r of 340 reached an angle on the south; and at remity 1270, there was a perpendicular of 530 on le. Ans. 7 ac. 3 ro. 11 per. 4 yds. 6¼ feet.



hexagon are given the sides C 498, CD 620, DE 580, nd AF 492 links, and the C 918, CE 1048, and AE

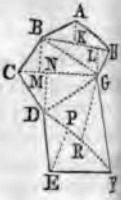
! ro. 9 per. 23 yds. 8·413 feet.



ıeptagon are given the sides ; 456, CD 572, DE 640, EF 3, and GA 386, and the dia- 540, AD 864, AE 630, and s.

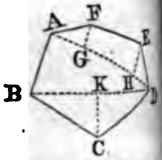
. 6 ac. 1 ro. 34 per. 18·3 yds.

7. In an octagon, the diagonals are BH 956, BG 874, GC 1078, GD 1178, and DF 1240 links; the sides AB 620, and DE 830; and the perpendiculars AK 326, GL 520, both on BH; those on GC are BM 610, DN 354; and on DF are EP 472, and GR 396 links.



Ans. 14 acres 2 roods 19 perches 13 yards.

8. Measured AB 538, and on diagonals from its extremities AG 324, and the perpendicular GF 260, AH 960, and the perpendicular HE 300; the whole diagonal AD 1240. And on the diagonal BD measured BK 460, and the perpendicular CK 350; the whole BD 1310 links. Ans. 8 acres 38 perches 5 yards.



9. The diagonals are AE 810, AC 930, CE 520; on AE at 245 is perpendicular GL 65, at 440 is perpendicular FM 198, on AC at 300 is perpendicular BN 189, on EC at 400 is perpendicular DP 125 links, all exterior.

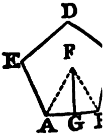
Ans. 4 acres 1 perch 1 yard 4·58 feet.

Prob. XII. To find the area of a regular polygon.

Rule. Multiply half the perimeter by the perpendicular dropt from the centre upon one of the sides.

For the polygon may be divided, by drawing lines from the centre to its angles, into as many triangles as it has sides, having equal bases and perpendiculars.

1. Required the area of the regular pentagon ABCDE, of which the side AB is 250 feet, and the perpendicular from the centre FG 172·05 feet.



$$\begin{array}{r} 172{\cdot}05 \\ 125 = 250 \times \frac{1}{2} \\ \hline 21506{\cdot}25 \\ 5 \\ \hline \end{array}$$

Ans. 107531·25 square feet.

Note. The perpendicular may be found from the side by trigonometry; for 360° divided by twice the number of

give the angle AFG, and its cotangent multiplied by AG gives FG the perpendicular.

2. What is the area of a regular octagon, the side 237 feet, the perpendicular is found to be 286·084?

Ans. 271207·63 square feet.

3. What is the area of a regular hexagon, the side 356 yards, the perpendicular 308·305? Ans. 329269·74 yards.

4. What is the area of a regular heptagon, the side 237 links? Ans. 2 acres 6 perches 17 yards 5·23 feet.

5. What is the area of a regular nonagon, the side 147 inches? Ans. 103 yards 95 inches.

6. What is the area of a regular decagon, the side 243 feet?

Ans. 10 acres 1 rood 28 perches 24 yards 6·4 feet.

RULE II. Multiply the square of the side by the multiplier corresponding to the figure in the following Table: the product will be the area.

Names.	No. of sides.	Angle centre.	Angle FAG.	Perpendiculars.	Multipliers.
Equilateral triangle,	3	120°	30°	0·2886752	0·4330127
Square, . . .	4	90	45	0·5000000	1·0000000
Pentagon, . . .	5	72	54	0·6881910	1·7204774
Hexagon, . .	6	60	60	0·8660254	2·5980762
Heptagon, . . .	7	$51\frac{3}{7}$	$64\frac{2}{7}$	1·0382607	3·6339124
Octagon, . . .	8	45	$67\frac{1}{2}$	1·2071068	4·8284272
Nonagon, . . .	9	40	70	1·3737387	6·1818242
Decagon, . . .	10	36	72	1·5388418	7·6942088
Undecagon, . .	11	$32\frac{8}{11}$	$73\frac{7}{11}$	1·7028439	9·3656411
Dodecagon, . .	12	30	75	1·8660254	11·1961524

The table is calculated by the first rule for polygons, of which the side is 1; and regular polygons being similar, are as the squares of their sides, (Appendix, Prop. 20, Cor. 3,) which gives the rule.

7. Required the area of a regular heptagon, of which the side is 327 feet.

$$\begin{array}{rr} \text{Tabular multiplier} = & 3·6339124 \\ & 327 \\ \hline & 1188·2893548 \\ & 327 \\ \hline \end{array}$$

Ans. 388570·6190196 square feet, = 8 ac. 3 ro. 27 per. $7\frac{3}{4}$ yds.

8. What is the area of an equilateral triangle, the side 436 yards? Ans. 82313·98 yards, = 17 ac. 1 per. 3·73 yds.

9. What is the area of a regular dodecagon, the side 254 poles? Ans. 4514 acres 2 roods 10 perches 29·29 yards.

10. What is the area of a regular undecagon, the side 27 yards? Ans. 1 acre 1 rood 25 perches 21 yards 2·7 feet.

11. What is the area of a regular decagon, the side 197 inches? Ans. 7 perches 18 yards 5 feet 128·55 inches.

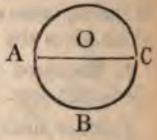
12. What is the area of a regular nonagon, the side 254 feet? Ans. 9 acres 24 perches 28 yards.

OF THE CIRCLE.

Prob. XIII. Given the diameter of a circle, to find the circumference.

Rule. Multiply the diameter by $3\frac{1}{7}$, or by 3·1416; or, if greater accuracy be required, by 3·141592653, &c.

Note. It will be shown in the Appendix, Prop. 77, Ex. 2, that the arc, of which t is the tangent, is $= t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7$, &c. If $t = \frac{1}{2}$, the length of the arc is $\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7}$, &c. $=$ ·463647609000807, &c.; and if $t = \frac{1}{3}$, the length of the arc will be $\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7}$, &c. $=$ ·321750554396641, &c.; and the sum of these two arcs is $=$ ·785398163397448, &c., and the tangent of their sum is $\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$, which is the tangent of 45°. Having thus found the length of the arc of 45°, multiply it by 4, and the product 3·14159265358979З, &c. is the length of the arc of 180° when the radius is 1, or it is the circumference when the diameter is 1.



1. Required the circumference of the circle of which the diameter is 356 yards.

	356	3·1416	3·1415926536
	$3\frac{1}{7}$	356	356
Ans.	1118·8	1118·4096	1118·4069846816

2. Required the circumference of the circle, of which the diameter is 628 links.

Ans. 3·1416 × 628 = 1972·9248 links, = 1 furlong 38 poles 5 yards 1·56 inches.

3. Required the circumference of a circle, of which the diameter is 7958 miles.

Ans. 25000·79434 miles, = 25000 m. 6 fur. 14 pol. 1 yd.

4. Required the circumference of a circle, of which the radius is 512 feet. Ans. 4 fur. 34 poles 5 yards 1 foot.

5. Required the circumference of a circle, of which the radius is 157 inches. Ans. 4 poles 5 yards 1 foot 2·46 inches.

6. Required the circumference of a circle, of which the radius is 38 poles. Ans. 5 fur. 38 poles 4 yards 6·79 inches.

Prob. XIV. Given the circumference of a circle; to find the diameter.

Rule. Divide the circumference by 3·1416, or multiply it by ·318309886.

1. Required the diameter of the circle, of which the circumference is 758 yards.

$$7580000 \div 31416 = 241{\cdot}278$$
$$31831 \times 758 = 241{\cdot}2789$$

Ans. 1 furlong 3 poles $4\frac{3}{4}$ yards.

2. Required the diameter of the circle, of which the circumference is 984 links.

Ans. 313·21693 links, = 12 poles 2 yards $2\frac{3}{4}$ feet.

3. Required the diameter of the circle, of which the circumference is 24855·43 miles. Ans. 7911·73 miles.

4. Required the diameter of the circle, of which the circumference is 398 ells. Ans. 126 ells 25 inches.

5. Required the diameter of the circle, of which the circumference is 928 poles. Ans. 7 fur. 15 poles 2 yds. 5·53 inches.

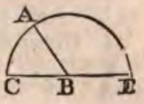
6. Required the diameter of the circle, of which the circumference is 1043 feet. Ans. 20 poles 1·997 feet.

Prob. XV. Given the radius and the number of degrees in an arc of a circle; to find the length of the arc.

Rule. Find the circumference, multiply it by the degrees, and divide by 360°.

Or multiply the radius by the number of degrees in the arc, and by ·0174533.

1. Required the length of an arc AC of 57°, in a circle of which the radius AB is 38 feet.



·0174533	3·1416
57	38
·9948381	119·3808
38	19
Ans. 37·8038478	60 | 226·82352
	37·80392

2. Required the length of an arc of 19° 37′, the radius being 98 yards. Ans. $\cdot01745 \times 19\cdot617 \times 98 = 33\cdot553$ yards.

3. Required the length of an arc of 134° 18′, the radius 9 feet 4 inches. Ans. 21·877 feet.

4. Required the length of an arc of 83° 24′, radius 32 poles. Ans. 1 furlong 6 poles 3 yards 6·72 inches.

5. Required the length of an arc of 150°, radius 19 ells. Ans. 8 falls 1 ell 27·45 inches.

6. Required the length of an arc of 17° 50′, radius 178 miles. Ans. 55 miles 3 furlongs 8 poles $4\frac{1}{2}$ yards.

PROB. XVI. Given the chord of an arc, and its height, or the versed sine of its half; to find the diameter.

RULE. Divide the square of half the chord by the height, and the quotient added to the height will be the diameter.

1. Given the chord BD 287, and the height CE 78 feet; to find the diameter AC.

$$\begin{array}{r} 2\,|\,287 \\ \hline 143\cdot5 \\ 143\cdot5 \\ \hline 78)\,20592\cdot25 \\ \hline 264 \\ 78 \\ \hline \end{array}$$

Ans. 342 diameter.

2. Given the chord 178, and height 257 yards. Ans. 287·821 yards.

3. Given the chord 843, height 648 links. Ans. 922·17 links, = 36 poles 4 yards 2 feet $7\frac{1}{2}$ inches.

4. Given the chord 40, height 12 yards. Ans. $45\frac{1}{3}$ yards.

5. . . . . . . 560, height 45 links. $1787\frac{8}{9}$ links.

6. . . . . . . 325, versed sine 78 ells. 416·54 ells.

PROB. XVII. Given the chord of an arc, and its height; to find the length of the arc.

RULE. Find the diameter by Prob. XVI.; then, as the diameter is to the chord, so is radius to the sine of half the angle measured by the arc, from which find the length of the arc by Prob. XV.

1. Required the length of the arc, of which the chord is 326, and its height 97 feet.

$163^2 \div 97 = 273{\cdot}90722$; and the diameter is $370{\cdot}90722$, and the radius $185{\cdot}45361$.

$326 + R.$	log. 12·5132176
370·90722	log. 2·5692652
Sin. 61° 30′ 47·2″	log. 9·9439524
2	

$123° \; 1' \; 34{\cdot}4'' = 123{\cdot}0262°$, the angle of the sector.

Ans. $185{\cdot}45361 \times 123{\cdot}0262 \times {\cdot}0174533 = 398{\cdot}2084$ the arc.

2. Required the length of the arc, of which the chord is 496, and the height 654 links. Ans. 1807·787 links.

3. Required the length of the arc, of which the chord is 126, versed sine 14 inches. Ans. 130·10809 inches.

4. Required the length of the arc, of which the chord is 78, versed sine 13 yards. Ans. 83·655 yards.

BY APPROXIMATION. Divide the height by half the chord, and square the quotient. To 3 times this square add 15, and to the sum add 10 times the square. Then as the former sum to the latter, so is the chord to the arc nearly. See Appendix, Prop. 77, Ex. 2.

Otherwise, having found the square as before: As $\frac{5}{9}$ of the square $+ 1$ to $\frac{1}{8}$ of it $+ 1$, so is $\frac{1}{5}$ of it to a fourth number. Subtract this number from 1, multiply the remainder by the square, and to the product add $1{\cdot}5$: this sum, multiplied by $\frac{2}{3}$ of the chord, will produce the arc very nearly. See Appendix, Prop. 77, Ex. 2.

5. Required the length of the arc, of which the chord is 40, and the height 6 feet.

$\frac{6}{20} = {\cdot}3$, and ${\cdot}3 \times {\cdot}3 = {\cdot}09$, the square to be used: then $3 \times {\cdot}09 + 15 = 15{\cdot}27 : 15{\cdot}27 + {\cdot}9 = 16{\cdot}17 :: 40 : 42{\cdot}358$ feet the arc.

By the second approximation, ${\cdot}09 \times \frac{5}{9} + 1 : {\cdot}09 \times \frac{1}{8} + 1 :: {\cdot}09 \times \frac{1}{5} : {\cdot}0173357$, and $(1 - {\cdot}0173357) \times {\cdot}09 + 1{\cdot}5 = 1{\cdot}58843979$, and $1{\cdot}58843979 \times \frac{2}{3} \times 40 = 42{\cdot}35843$.

6. Required the length of the arc, of which the chord is 184, and the height 34 feet. Ans. 200·3217 feet.

7. Required the length of the arc, of which the chord is 246, and the height 534 links.

NOTE. When the height is greater than the chord, find the diameter, and from it subtract the height, to get the height of the other segment; find its arc, and subtract it from the circumference.

Ans. 1512·0056 links, = 1 fur. 20 poles $1\frac{1}{3}$ yards.

8. Required the length of the arc, of which the chord is 128, height 216 feet. Ans. 602·7928 feet.

9. Required the length of the arc, of which the chord is 76, height 22 links. Ans. 91·98254 links.

PROB. XVIII. Given the radius and the circumference of a circle; to find its area.

RULE. Multiply the radius by half the circumference: the product is the area.

The area of a semicircle, or of a quadrant, is a half or a fourth of the area of a circle.

NOTE. The circle is the limit of the polygons inscribed in it and described about it, and the circumference is the limit of their perimeters, and the radius the limit of the perpendiculars, and any polygon is = perpendicular $\times \frac{1}{2}$ perimeter, therefore the circle is = radius $\times \frac{1}{2}$ circumference. See Appendix, Prop. 46.

1. Required the area of the circle, of which the radius is 75, and the circumference 471·24 yards.

$471{\cdot}24 \times \frac{1}{2} \times 75 = 17671{\cdot}5$ square yards, = 3 acres 2 roods 24 perches $5\frac{1}{2}$ yards.

2. Required the area of the circle, of which the diameter is 10, and the circumference 31·416. Ans. 78·54.

3. Required the area of the circle, of which the diameter is 7958, circumference 25001 miles. Ans. $49739489\frac{1}{2}$ miles.

4. Required the area of the circle, of which the diameter is 223, and the circumference 700 yards.

Ans. 8 acres 10 perches $2\frac{1}{2}$ yards.

5. Required the area of the circle, of which the diameter is 751, and the circumference 2485 feet.

Ans. 10 acres 2 roods 33 perches 21 yards $5\frac{1}{2}$ feet.

6. Required the area of the circle, of which the diameter is 169, and the circumference 532 inches.

Ans. 17 yards 3 feet 13 inches.

PROB. XIX. Given the radius or diameter of a circle; to find the area.

RULE. Multiply the square of the radius by 3·1416, or that of the diameter by ·7854.

NOTE. If R = radius, and D = diameter, then $3\cdot1416 \times R = \frac{1}{2}$ circumference; therefore $3\cdot1416 \times R^2 = \frac{1}{4} \times 3\cdot1416 \times D^2 = \cdot7854\ D^2$, will be the area. See Appendix, Prop. 47, Cor. 2.

1. Required the area of a circle, of which the radius is 78 feet.

Ans. $3\cdot1416 \times 78 \times 78 = 19113\cdot4944$ square feet, = 1 rood 30 perches $6\frac{1}{4}$ yards.

2. Required the area of a circle, of which the diameter is 234 yards.

Ans. $234 \times 234 \times \cdot7854 = 43005\cdot3624$ square yards, = 8 acres 3 roods 21 perches 20 yards.

3. Required the area of a circle, of which the diameter is 563 links.

Ans. 248947·4526 square links, = 2 acres 1 rood 38 perches 9 yards 5 feet.

4. Required the area of a circle, of which the diameter is 7·5 feet. Ans. 44·17875 feet.

5. Required the area of a circle, of which the radius is 193 yards. Ans. 24 acres 28 perches 14 yards.

6. Required the area of a circle, of which the diameter is 9 feet 6 inches. Ans. 7 yards 7 feet 127 inches.

7. Required the area of a circle, of which the radius is 59 poles. Ans. 68 acres 1 rood 15 perches 27 yards.

PROB. XX. Given the circumference of a circle; to find the area.

RULE. Divide the square of half the circumference by 3·1416.

Or multiply the square of the circumference by ·0795775 to get the area.

1. Required the area of a circle, of which the circumference is 1284 yards.

$$642 = 1284 \times \tfrac{1}{2}$$
$$\underline{642}$$

Ans. 3·1416)412164(131195·569 square yards, = 27 acres 17 perches $1\frac{1}{3}$ yards.

2. Required the area of a circle, of which the circumference is 1386 links.

Ans. 152868 square links, = 1 ac. 2 ro. 4 per. 17·8 yds.

3. Required the area of a circle, of which the circumference is 73 feet 8 inches.

Ans. 431·84942 square feet, = 1 perch $17\frac{3}{4}$ yards.

4. Required the area of a circle, of which the circumference is 625 yards. Ans. 6 acres 1 rood 27 perches 18·2 yards.

5. Required the area of a circle, of which the circumference is 1448 feet. Ans. 3 ac. 3 ro. 12 per. 25 yds. 8·46 feet.

6. Required the area of a circle, of which the circumference is 627 poles. Ans. 195 acres 2 roods 4·2 perches.

7. Required the area of a circle, of which the circumference is 178 inches. Ans. 1 yard 8 feet $73\frac{1}{2}$ inches.

Prob. XXI. To find the area of a sector of a circle.

Rule I. If the length of the arc be known, multiply half the arc by the radius.

Rule II. If the angle of the sector be given, find the length of the arc, and work as before. Or find the area of the circle: then, as 360° to the angle of the sector, so is the area of the circle to the area of the sector.

1. Required the area of a sector, of which the arc is 79, and the radius of the circle 47 yards.

$$\begin{array}{r} 39{\cdot}5 = 79 \times \frac{1}{2} \\ 47 \\ \hline \end{array}$$

Ans. 1856·5 square yards, = 1 ro. 21 per. $11\frac{1}{4}$ yds.

2. Required the area of a sector, of which the arc is 17 feet 5 inches, the radius 22 feet.

Ans. 191·583 square feet, = 21 yards 2·583 feet.

3. Required the area of a sector, of which the angle is 127° 16′, the radius 133 feet.

Ans. 19645·6 square feet, = 1 rood 32 perches 4·845 yards.

The area of the circle is 55571·63245; and this, multiplied by $127\frac{4}{15}$, and divided by 360, gives 19645·601175.

4. Required the area of a sector, of which the angle is 137° 20′, the radius 456 links.

Area = 2 acres 1 rood 38 perches 21·95 yards.

5. Required the area of a sector, of which the angle is 27°, the radius 97 miles. Ans. 2216·95 miles.

6. Required the area of a sector, of which the arc is 156 yards, the radius 478 feet. Ans. 3 ro. 16 per. 28 yds. 6 feet.

Prob. XXII. To find the area of a segment.

Rule I. Find the area of the sector which has the same arc with the segment, and from it subtract the area of the triangle contained by the chord and the radii drawn to its extremities, when the segment is less that a semicircle. Otherwise, add these areas, and the remainder or the sum will be the area of the segment.

1. Required the area of the segment ABC, of which the height BD is 6, and the diameter of the circle BE 32 feet.

$\sqrt{26\times 6} \div 16 = 12{\cdot}49 \div 16 = {\cdot}780625 = \sin. 51{\cdot}3175°$, and $(51{\cdot}3175 \div 180) \times 3{\cdot}1416 \times 256 = 229{\cdot}289$ sector, and $229{\cdot}289 - 12{\cdot}49 \times 10 = 104{\cdot}389$ square feet the segment.

2. Required the area of the segment, of which the chord is 12, and the diameter 36 yards.

$\frac{6}{18} = {\cdot}33333$ the sine of $19{\cdot}47122°$. Ans. 8·283 yards.

3. Required the area of the segment, of which the chord is 20, and the height 2.

The diameter is 52, the angle 45·2397°. Ans. 26·8786995.

4. Required the area of the segment, of which the height is 18, and the radius 56 yards. Ans. 33 perches 25$\frac{3}{4}$ yards.

5. Required the area of the segment, of which the chord is 257, the diameter 824 feet. Ans. 13 perches.

6. Required the area of the segment, of which the chord is 540, and the height 29 links. Ans. 16 per. 22 yds. 4$\frac{3}{4}$ feet.

Rule II. By a Table of Segments. Divide the height by the diameter. Look in the table for the quotient in the column of versed sines, and take out the number on the right hand of it in the column of areas, and multiply it by the square of the diameter, and the product will be the area of the segment.

Note. If the height be greater than the radius, subtract it from the diameter to get the height of the other segment. Find the area of this segment by the rule, and subtract it from the area of the circle to get the area of the segment required.

7. Required the area of the segment, of which the height is 18, and the diameter of the circle 48.

Ans. $\frac{18}{48} = {\cdot}375$, opposite to which is ·26901365, and $48 \times 48 \times {\cdot}26901365 = 619{\cdot}80745$ the area.

8. Required the area of the segment, of which the height is 236, and the diameter 432 links.

Ans. $\frac{432 - 236}{432} = {\cdot}4537$, opposite to which is ·34646534 the other segment, and ${\cdot}78539816 - {\cdot}34646534 = {\cdot}43893282$ the segment required from the table. Wherefore $432^2 \times {\cdot}43893282 = 81915{\cdot}399234$ links the area, = 3 roods 11 per, 2 yards.

9. Required the area of the segment, of which the chord is 354, the height 18 feet. Ans. 15 per. 19 yds. 3·63 feet.

10. Required the area of the segment, of which the height is 26, and the diameter 298 yards.

Ans. 2 roods 18 perches 5 yards 7·34 feet.

11. Required the area of the segment, of which the radius is 125, and the height 36 links. Ans. 6 perches 29 yards.

By Approximation. To the chord add $\frac{1}{3}$ of the chord of half the segment, and multiply the sum by $\frac{2}{5}$ of the height: the product will be the area nearly.

More accurately. Divide the height by half the chord, and square the quotient; and as 5 times the square + 11 to 4 times the square + 33, so is $\frac{1}{21}$ of the square to a fourth number. Subtract this number from 1, and multiply the remainder by the square, and to the product add 5; then multiply this sum by the chord and by the height, and $\frac{2}{15}$ of the product will be the area very nearly. See Appendix, Prop. 78, Ex. 2.

12. Required the area of the segment, of which the chord is 50, and the height 3.

Ans. $\surd(25^2+3^2)=25{\cdot}1794$ the chord of $\frac{1}{2}$ the segment; then $(50+25{\cdot}1794\times\frac{4}{3})\times{\cdot}4\times 3=100{\cdot}287$ the area nearly.

By the second method, $\frac{3}{25}={\cdot}12$ and ${\cdot}12^2={\cdot}0144$ the square, and $5\times{\cdot}0144+11=11{\cdot}072 : 4\times{\cdot}0144+33=33{\cdot}0576 :: \frac{1}{21}\times{\cdot}0144={\cdot}0006\dot{8}5714\dot{2} : {\cdot}0020473328$ the fourth number: then $(1-{\cdot}0020473328)\times{\cdot}0144+5=5{\cdot}014370518408$; and this, multiplied by $50\times3\times\frac{2}{15}$, gives $100{\cdot}287410368$ the area.

13. Required the area of the segment, of which the chord is 178, and the height 14 inches. Ans. 11 feet $85\frac{1}{2}$ inches.

14. Required the area of the segment, of which the chord is 560, the height 29 poles. Ans. 67 acres 3 roods 9·8 perches.

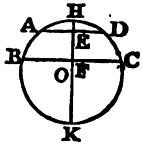
Note. If the height be greater than half the radius, find the area of the segment subtended by the chord of half the arc, and to its double add the area of the triangle contained by the chords. To find the height of this small segment: Having found the chord of half the arc for the chord of it, multiply it by half the chord of the given segment, and subtract the product from the square of the chord of half the arc: the remainder, divided by twice the height, will give the height of the small segment.

Required the area of the segment, of which the chord is
id the height 32 inches.
3437·474107 square inches, = 2 yds. 5 feet 125½ inches.
Required the area of the segment, of which the chord
nd the height 48 yards.
ns. 2886·325466 square yards, = 2 ro. 15 per. 12·6 yds.
Required the area of the segment, of which the chord
nd the height 15 poles.
ns. 303·5307427 sq. poles, = 1 ac. 3 ro. 23 per. 16 yds.
Required the area of the segment, of which the chord
and the height 152 feet.

Ans. 2 roods 38 perches 10 yards 6¼ feet.

B. XXIII. To find the area of a zone, or of a
the circle intercepted between two parallels.

E. Find the areas of the segments cut off by the chords,
ir difference will be the area of the zone.
nd the area of the segment cut off by the straight line
the extremities of the chords, and the area of the
id formed by the chords; and the double of the seg-
lded to the trapezoid will be the area of the zone.



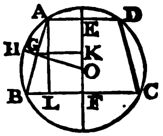
equired the area of the zone ABCD,
h the distance OE of the chord AD
e centre is 44, and the distance OF 13,
diameter HK 104 yards.

13) = 39 ÷ 104 = ·375 vers. sin. to seg. ·26901365
44) = 8 ÷ 104 = ·076923 ·02778038

Difference of segments, ·24123327
104^2 10816

Area of the zone, 2609·179048 yds.

Ans. 2 roods 6 perches 7·679 yards.



equired the area of a zone, of which
rds are AD 15 and BC 20, and their
EF 17½.
some preparation is requisite. Let
he centre, join AB, and draw OG
icular to AB, meeting the circle in
raw GK parallel to AD, and AL
to EF; then GK = ½(AE + BF) = 8¾, and BL =
AE = 2½. Also, AL : LB :: GK : KO = 1¼,

(Appendix, Prop. 18,) and $OF = FK - KO = 7\frac{1}{2}$. Now $OG^2 = OK^2 + KG^2$, (Appendix, Prop. 21, Cor. 2,) therefore $OG = 8{\cdot}838834765$; and $OB^2 = OF^2 + FB^2$, (Appendix, Prop. 21, Cor. 2,) therefore OB or $OH = 12{\cdot}5$, and $GH = 3{\cdot}661165$, which divided by 25 gives ·1464466 for the versed sine, for which the area is ·07134954; and this multiplied by 25^2, gives 44·5934625 the area of the segment AHB, and the trapezoid $ABCD = \frac{1}{2}EF \times (AD + BC) = 306{\cdot}25$, which, added to twice the segment, gives the zone 395·436925.

3. Required the area of a zone, having the parallel chords 96 and 60, and their distance 26 yards.

Ans. 2136·7528 square yards, = 1 ro. 30 per. 19¼ yds.

4. Required the area of a zone, the parallels each 36, and their distance 84 feet.

Ans. 6380·81726 square feet, = 23 per. 13 yds. 2 feet.

5. Required the area of a zone, the parallels 136 and 68, and their distance 248 feet.

Ans. 55655·2 square feet, = 1 ac. 1 ro. 4 per. 12 yds. 8·2 ft.

6. Required the area of a zone, the parallels 157 and 216; and their distance 128 yards.

Ans. 3 acres 34 perches 22 yards 7¼ feet.

7. Required the area of a zone, the parallels 247 and 194, and their distance 368 feet. Ans. 3 ac. 17 per. 23 yds. 6½ feet.

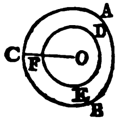
8. Required the area of a zone, the parallels 32 and 40, and their distance 72 inches. Ans. 33 feet 138 inches.

Prob. XXIV. To find the area of a ring contained by two concentric circles.

Rule. Multiply the sum of the diameters by their difference, and then by ·7854.

Note. If the circumferences or similar arcs of the circles be given, multiply half their sum by the difference of the radii: the product will be the area of the ring, or of the part of it contained by the similar arcs.

1. Required the area of the ring ABC–DEF, of which the diameters are 10 and 6, or OC 5 and OF 3.



$(10+6)(10-6) \times {\cdot}7854 = 50{\cdot}2656$ the area of the ring.

2. Required the area of the ring, of which the radii are 36 and 24 feet. Ans. 2261·952 square feet, = 8 perches 9½ yards.

3. Required the area of the ring, of which the radii are 10 and 6, and similar arcs 15 and 9.

Ans. $12 \times 4 = 48$, the area contained by the arcs.

4. Required the area of the ring, of which the radii are 157 and 128 yards. Ans. 5 ac. 1 ro. 18 per. 10 yds. 7·42 feet.

5. Required the area of the ring, of which the diameters are 246 and 228 inches. Ans. 46 feet 77 inches.

OF THE ELLIPSE.

Prob. XXV. To find the area of an ellipse.

Rule. Multiply one of the semiaxes by the other, and by 3·1416; or one of the axes by the other, and by ·7854.

Or if the circle upon either axis be given: As that axis is to the other, so is the circle to the ellipse, and so is any sector or segment of the circle to the sector or segment of the ellipse, which has the same chord perpendicular to the first-mentioned axis. See Appendix, Prop. 78, Ex. 3.

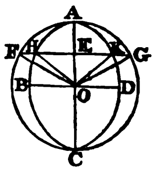
Note. If any two straight lines be drawn perpendicular to AC, and the points be joined in which they meet the circle and the ellipse, these trapezoids are to one another as EG to EK, and their number may be multiplied, until their sum either in the circle or ellipse shall be more nearly equal to it than by any given difference. Therefore the circle and ellipse which are their limits are in that ratio; that is, the circle is to the ellipse as EG to EK, or AC : BD, or as $AC^2 \times {\cdot}7854 : AC \times BD \times {\cdot}7854$.

1. Required the area of the ellipse ABCD, of which the semiaxes are OA 436, and OB 254 feet.

$3{\cdot}1416 \times 436 \times 254 = 347913{\cdot}3504$ square feet, = 7 acres 3 roods 37 perches 27 yards 7 feet.

2. Required the area of an ellipse, of which the axes are 526 and 354 inches. Ans. 112 yards 7 feet 84 inches.

3. Required the area of the sector OHAK of an ellipse, the chord HK being perpendicular to the greater axis AC; the axes AC 72 and BD 54, and the versed sine AE 18. The angle FOG is 120°. The circle = 4071·50408, and $\frac{1}{3}$ of it $\times \frac{3}{4} = 1017{\cdot}87602$ the area of the sector.



4. Required the area of the segment of an ellipse, the chord being perpendicular to the less axis, the versed sine 12, and the axes 80 and 60 yards.

Ans. 536·75424 square yards, = 17 perches $22\frac{1}{2}$ yards.

5. Required the area of the segment of an ellipse, the chord being perpendicular to the greater axis, the height 25 feet, and the axes 156 and 120 feet.

Ans. 5 perches 17 yards $7\frac{3}{4}$ feet.

6. Required the area of the segment of an ellipse, the chord being perpendicular to the less axis, the height 110, and the axes 246 and 180 yards.

Ans. 4 acres 2 roods 16 perches 3 yards $8\frac{1}{4}$ feet

PROB. XXVI. To find the circumference of an ellipse.

RULE. Add the squares of the two axes, and take the square root of half the sum, and to the half of this root add a fourth of the sum of the axes, and then multiply by 3·1416: the product will be the circumference nearly. See Appendix, Prop. 77, Ex. 3.

1. Required the circumference of the ellipse, of which the axes are 24 and 18.

$\sqrt{\frac{24^2+18^2}{2}} = 21{\cdot}2132$, and $\frac{24+18}{2} = 21$, and $(21{\cdot}2132+21) \div 2 \times 3{\cdot}1416 = 66{\cdot}3085$ the circumference.

2. Required the circumference of the ellipse, of which the axes are 60 and 40 feet.

Ans. 158·6351 feet, = 9 poles 3 yards 1 foot 1·6 inches

3. Required the circumference of the ellipse, of which the axes are 256 and 196 feet. Ans. 713·1156 feet

4. Required the circumference of the ellipse, of which the axes are 320 and 240 yards. Ans. 884·1133 yards

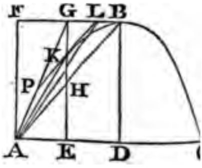
5. Required the circumference of the ellipse, of which the axes are 16·6 and 12·8 inches. Ans. 46·3736 inches

6. Required the circumference of the ellipse, of which the axes are 27 and 18 poles. Ans. 1 furlong 31 poles 2 yards

OF THE PARABOLA.

PROB. XXVII. To find the area of a parabola.

RULE. Multiply the base by the perpendicular height, and $\frac{2}{3}$ of the product will be the area. See Appendix, Prop. 78, Ex. 1.



NOTE. If EG bisect AD, the triangle AFG $= \frac{1}{2}$AFB, or it is $> \frac{1}{2}$ trilineal AFBK. Also, since GK = KH, the triangle PLG $= \frac{1}{2}$ALG, or $> \frac{1}{2}$ trilineal AGBK; and every triangle thus formed cuts off more than the half of what was left by the preceding; therefore the trilineal AFBK is the limit of the sum of the triangles. Now the triangle AFG $= \frac{1}{4}$FD, and the triangle GPL $= \frac{1}{4}$AGB, or of AFG, and so on; therefore

the sum of them is $FD \times \left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}, \&c.\right)$, and the limit of this geometrical series is (Prop. 3, Cor. 3,) $FD \times \frac{1}{4-1} = \frac{1}{3}FD = \frac{1}{3}BD \times AD$, and therefore $AKBD = \frac{2}{3}FD$.

1. Required the area of the parabola ABC, of which the base AC is 54, and the height BD 36 feet.

$$\tfrac{2}{3} \times 54 \times 36 = 1296 \text{ feet area.}$$

2. Required the area of the parabola, of which the base is 42, and the height 63 yards.

Ans. 1764 yards, = 1 rood 18 perches $9\frac{1}{2}$ yards.

3. Required the area of the parabola, of which the base is 482, and the height 320 feet.

Ans. 2 acres 1 rood 17 perches 20 yards $8\frac{5}{12}$ feet.

4. Required the area of the parabola, the base 126, and the height 210 inches. Ans. 13 yards 5 feet 72 inches.

5. Required the area of the parabola, the base 67, and the height 98 yards. Ans. 3 roods 24 perches $21\frac{1}{3}$ yards.

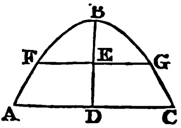
6. Required the area of the parabola, the base 16, and the height 12 poles. Ans. 3 roods 8 perches.

Prob. XXVIII. To find the area of a frustum of a parabola.

A Frustum is what remains after a part has been cut off from the top by a line parallel to the base.

Rule. Find a third proportional to the sum of the bases, and one of them, to which add the other base: the sum, multiplied by two-thirds of the height, gives the area. See Appendix, Prop. 78, Ex. 1.

1. Required the area of the frustum of a parabola, of which the bases are 64 and 32, and the height 26 feet.



$$64+32 : 32 :: 32 : 10\tfrac{2}{3}$$
$$\begin{array}{r} 64 \\ \hline 74\frac{2}{3} \\ \frac{2}{3}\times 26 = 17\frac{1}{3} \\ \hline \end{array}$$

Ans. Area $1294\frac{2}{9}$ feet, = 4 perches 22·8 yards.

2. Required the area of the frustum of a parabola, of which the bases are 16 and 54, and the height 46 yards.

Ans. 1768·15238 square yards, = 1 ro. 18 per. 13·65 yds.

3. Required the area of the frustum of a parabola, of which the bases are 364 and 186, and the height 280 feet.

Ans. 1 acre 3 roods 12 perches 21 yards 2 feet.

4. Required the area of the frustum of a parabola, of which the bases are 424 and 268, and the height 318 inches.

Ans. 2 perches 25 yards 7 feet 75·8828 inches.

5. Required the area of the frustum of a parabola, of which the bases are 63 and 22, and the height 44 poles.

Ans. 12 acres 2 roods 15 perches.

6. Required the area of the frustum of a parabola, of which the bases are 18 and 12, and the height 20 yards.

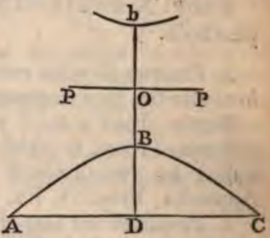
Ans. 10 perches $1\frac{1}{2}$ yards.

PROB. XXIX. To find the area of a hyperbola.

RULE. Multiply half the base by the semitransverse axis, and its distance from the centre by the semiconjugate, and divide the sum of the products by the product of the two semiaxes, and take the hyperbolic logarithm of the quotient, and multiply it by the product of the semiaxes, and subtract the product from the product of half the base by its distance from the centre: the remainder will be the area. See Appendix, Prop. 78, Ex. 4.

NOTE. The hyperbolic logarithm is got by multiplying the common logarithm by 2·30258509.

1. Required the area of the hyperbola ABC, of which the base AC is 24, and the altitude BD 10, and the transverse axis Bb 30, and the conjugate Pp 18 feet.



$\frac{12\times15+25\times9}{15\times9}=3$, of which the logarithm $0\cdot4771212\times2\cdot30258509=1\cdot0986123$ the hyperbolic logarithm of 3; and this logarithm, multiplied by 15×9, gives 148·3126605, which, taken from 25×12, leaves 151·6873395 the area, = 16 yards 7 feet 99 inches.

2. Required the area of the hyperbola, of which the base is 208, the height 70, and the transverse semiaxis 105 yards.

$\sqrt{((210+70)\times70)} : 104 :: 105 : 78$ the semiconjugate.

Ans. 9202·365 square yards, = 1 ac. 3 ro. 24 per. $6\frac{1}{3}$ yds.

3. Required the area of the hyperbola, of which the base is 384, the height 250, and the axis 176 feet. Ans. 55686 feet.

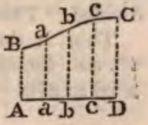
4. Required the area of the hyperbola, of which the base is 156, height 196, axis 248 yards. Ans. 18449·84 yards.

5. Required the area of the hyperbola, of which the base is 48, height 22, axis 36 inches. Ans. 647·2483 inches.

6. Required the area of the hyperbola, of which the base is 96, height 110, axis 124 poles. Ans. 6324·686 poles.

PROB. XXX. To find the area of a space bounded on one side by a curve line.

RULE. Let perpendiculars be erected upon the base, so numerous, that the part of the curve between any two nearest to one another shall differ very little from a straight line. Then add the perpendiculars at the extremities of the base, if there are any, and to half their sum add the rest of the perpendiculars. Multiply the sum by the base, and divide the product by the number of parts into which the base is divided by the perpendiculars: the quotient will be the area nearly.



1. Suppose the perpendiculars at the extremities of the base to be 10 and 16, and the other perpendiculars to be 11, 14, 16, and the base to be 20 feet.

$$
\begin{array}{r}
(10+16)\times\frac{1}{2}=13 \\
11 \\
14 \\
16 \\
\hline
54 \\
20 \\
\hline
4)1080 \\
\hline
\end{array}
$$

Ans. 270 square feet the area.

2. A curve-lined space meets the base at one of its extremities, and the perpendicular at the other extremity is 96, the other perpendiculars are 83, 70, 64, 51, 38, 25, and the base 325 links. What is the area? Ans. $17596\frac{3}{7}$ square links.

3. An offset meets the base at both extremities, the base is 252 links, and the perpendiculars are 24, 36, 42, 54, 67, 76, 58, 49, 33, and 19. Required the area.

Ans. $10492\frac{4}{11}$ square links.

4. Perpendiculars were raised from the base to a curve; those at the ends were 364 and 578, the others were 396, 418, 453, 512, and 554 links, the base 1260 links.

Ans. 5 acres 3 roods 22 perches 4 yards 3·2 feet.

5. A curve meets the base at one extremity, the base is 2364, the perpendicular at the other extremity 758, and the others are 642, 587, 524, 432, 417, and 335 links.

Ans. $1119860\frac{4}{7}$ links, = 11 acres 31 perches 23·5 yards.

NOTE 1. This rule supposes the figure to be divided into trapeziods, and would be exact if the breadths of the trapeziods were all equal. But the common rule is to add all the perpendiculars, and to multiply by the base, and divide by the

number of perpendiculars; which is not much easier, and gives the answer sometimes considerably erroneous. Thus the third example would come to 11541·6.

Note 2. If the distances between the perpendiculars be equal, the curvature, if single, may be considered as parabolical. And taking care to have an odd number of perpendiculars, add the first and last perpendiculars into one sum, the second, fourth, &c. into another, and all the rest into a third sum; then add the first sum, twice the third, and four times the second sum together, multiply this by the base, and divide by three times the number of parts into which the base is divided. The quotient is the area.

Thus, in the first example, the first sum is 26, the second 27, and the third 14; therefore $(26+4\times27+2\times14)\times\frac{1}{12}\times20=270$.

MENSURATION OF SOLIDS.

THE SOLID CONTENT of a body is the number of cubical inches, feet, &c. which the body contains.

A CUBICAL INCH is a solid contained by six square inches; or it is a solid, of which the length, breadth, and thickness are each of them an inch. And the same is to be understood respecting a cubical foot, yard, &c.

TABLE OF CUBICAL MEASURE.

1728	cubical inches	make	1	cubical foot.
27	feet		1	yard.
$166\frac{3}{8}$	yards		1	pole.
64000	poles		1	furlong.
512	furlongs		1	mile.

NOTE. 231 cubical inches make a wine gallon, 282 cubical inches make an ale gallon, 2150·42 cubical inches make a malt bushel, and 104·2 such inches make a Scotch pint.

All these measures are now laid aside by act of parliament, and the only legal standard for measuring both liquid and dry goods is declared to be the imperial gallon, containing 10 pounds avoirdupois weight of distilled water weighed in air at the temperature of 62 degrees of Fahrenheit's thermometer, the barometer being at 30 inches; each avoirdupois pound containing 7000 troy grains. It is declared that this gallon is to contain 277·274 cubic inches of rain water. A pint is the eighth part of a gallon, 8 gallons make a bushel of 4 pecks, and 8 bushels make a quarter. Hence a wine gallon is 0·8331109 imperial gallon, an ale gallon 1·017045 imperial gallon, a Winchester bushel 0·969448 imperial bushel, a Scotch wheat firlot 0·998256 imperial bushel, a Scotch barley firlot 1·4562794 imperial bushel, and a Scotch pint 0·375814 imperial gallon.

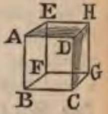
PROB. I. To find the surface of a prism.

A PRISM is a solid of which the ends are equal, similar, and parallel rectilineals, and the other sides are parallelograms.

NOTE. If the ends be parallelograms, the prism is called a *Parallelopiped;* and if all its sides be squares, it is called a *Cube.*

RULE. Find the area of one of its ends, and to its double add the sum of the areas of the parallelograms.

1. Required the surface of a cube, upon a line of 37 inches.



$$\begin{array}{r} 37 \\ 37 \\ \hline 1369 \\ 6 \\ \hline \end{array}$$

Ans. Surface 8214 square inches.

2. Required the surface of a rectangular parallelopiped, of which the length is 11 feet, and each side of the base 27 inches. Ans. 109 feet 18 inches.

3. Required the surface of a pentagonal prism, the length 14 feet, and each side of the base 33 inches. Ans. 218·52 feet.

TO FORM A PRISM WITH PASTEBOARD.

Let ABCD be one of the parallelograms of which the sides are compounded, AB the length, and AD a side of the base. Extend AD and BC, and make the parallelograms DK, AL, FM, &c. each equal to AC, and upon AD and BC make figures equal to the bases.

Then if the figure thus formed be cut out of the pasteboard, and folded at the sides of the parallelograms till they meet, the prism will be formed, and its surface is the figure cut out.

4. Required the surface of a chest, of which the length is 7 feet 8 inches, the breadth 4 feet 7 inches, and the depth 2 feet 9 inches. Ans. 137 feet 7 inches 10 parts.

5. Required the surface of a triangular prism, of which the length is 13 feet, and the sides of the base 23, 34, and 19 inches. Ans. 85·2241 square feet.

PROB. II. To find the solid content of a prism.

RULE. Find the area of one of the ends, and multiply it by the length or perpendicular height.

NOTE. If the height be one foot, the solid will contain as many cubical feet as there are square feet in the base; if the height be two feet, the solid will contain twice as many cubical feet; if the height be three feet, it will contain three times as many, and so on.

. Required the solid content of a triangular ism, of which the height is 9 feet, and each side he base 34 inches.

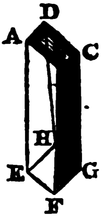
Tabular number 0·4330127
$34 \times 34 = 1156$ square inches.

500·5626812
9 feet.

144) 4505·0641308

Ans. Content 31·2851676 cubic feet.

2. Required the solid content of a rectangular tern, of which the length is 3 feet 2 inches, the adth 2 feet 8 inches, and the depth 2 feet 6 hes. Ans. 21 feet 1 inch 4 parts.



3. Required the solid content of a heptagonal ism, of which the length is 21 feet, and each le of the base 43 inches.

Ans. 979·86934 cubic feet.

4. Required the solid content of a pentagonal ism, the length 23 feet, and each side of the base 54 inches.

Ans. 801 cubic feet 539·739 cubic inches.

5. Required the solid content of a quadrilateral prism, the ngth 19 feet, the sides of the base 43, 54, 62, and 38, and e diagonal between the first and second 70 inches.

Ans. 306 cubic feet 81·976 inches.

Prob. III. To find the surface of a cylinder.

A Cylinder is a round solid of uniform thickness, of ich the bases are equal and parallel circles.

Rule. Multiply the circumference of the base by the ght: the product is the curve surface, to which add the as of the two bases. See Appendix, Prop. 79.

1. What is the curve surface of a cylinder, of which the gth is 16 feet, and the diameter of the base 27 inches?

3·1416
$2\frac{1}{4}$

7·0686
16

Ans. Surface 113·0976 square feet.

2. Required the whole surface of a cylinder 13 feet long, l having the circumference of its base 57 inches.

Ans. 65·3409 square feet.

3. Required the whole surface of a cylinder, the length 15 feet, and the radius of the base 23 inches. Ans. $24133\frac{3}{4}$ inches.

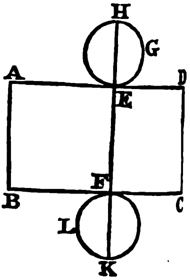
4. Required the curve surface of a cylinder, the length 15 feet, and the diameter of the base 33 inches.

Ans. 129 square feet 85 inches.

5. How often must a cylinder 5 feet 3 inches long, and the diameter of its base 21 inches, revolve, to roll an acre?

Ans. 1509·18 times.

TO FORM A CYLINDER WITH PASTEBOARD.



Find the circumference of the base, and make the rectangle ABCD, of which AD is the circumference, and AB the length of the cylinder; and draw EF parallel to AB, and make EH, FK, each the diameter of the base, and describe the circles EGH and FKL. The figure thus formed being cut out of the paper, and bended round, so that AB meet CD, will form the cylinder. The area of the figure is the surface of the cylinder.

Prob. IV. To find the solid content of a cylinder.

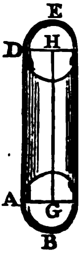
Rule. Find the area of the base, and multiply it by the perpendicular height or length.

Note. This is proved the same way as that of the prism.

1. Required the solid content of the cylinder, of which the length is 9 feet, and the circumference of the base 6 feet.

$$\begin{array}{r} \cdot0795775 \\ 36 \\ \hline 2\cdot86479 \\ 9 \\ \hline \end{array}$$

Ans. Content 25·7831 cubic feet.



2. Required the solid content of the cylinder, of which the length is 11 feet, and the diameter of its base 38 inches.

Ans. $\cdot7854 \times 3\frac{1}{6} \times 3\frac{1}{6} \times 11 = 86\cdot63398$ cubic feet.

3. Required the solid content of an oblique cylinder, the axis of which makes an angle of 75° with the base, the axis and the circumference of the base being each 20 feet.

Sin. $75° = \cdot9659258 \times 20 = 19\cdot318516$ the perpendicular height. Ans. 614·92768 cubic feet.

An upright cylinder 20 feet high, and the diameter of ase 3 feet, is cut by a plane parallel to the axis, and 12 s from it. Required the content of each of its segments.

Ans. 15·48741 and 125·88426 cubic feet.

Required the solid content of an upright cylinder 24 feet the base an ellipse, of which the axes are 32 and 24 s. Ans. 100 cubic feet 917½ inches.

Required the solid content of an oblique cylinder, of ı the axis inclines in an angle of 60°, the length 25 feet, he diameter of the base 30 inches.

Ans. 106 cubic feet 479½ inches.

Required the solid content of an oblique cylinder, of ı the length is 18 feet, and the base an ellipse, of which xes are 35 and 28 inches, the inclination is over the :r axis 56°. Ans. 79 cubic feet 1318 inches.

:OB. V. To find the surface of a pyramid.

PYRAMID is a solid which has a rectilineal figure for its and its sides are triangles which have a common vertex.

'LE. Find separately the area of the base, and the areas triangles which constitute its sides, and add them: the rill be the whole surface.

Required the surface of a triangular pyramid, of which side of the base is 32 inches, and the perpendicular from :rtex upon a side of the base 11½ feet.

	F.	I.		
	11	6		0·4330127
	1	4	$32^2 =$	1024
	15	4		144)443·405005
		3		
				3·0792 area of base.
es	46			46

Ans. Whole surface 49·0792 square feet.

What is the surface of a square pyramid, each side of the :8 inches, and the perpendicular upon a side from the : 9 feet? Ans. 47¾ square feet.

What is the surface of a pentagonal pyramid, the slant ɔdicular from the vertex 10 feet, a side of the base 26 ? Ans. 62·24335 square feet.

What is the whole surface of a triangular pyramid, of the slant height is 18 feet, and each side of the base 42 ? Ans. 99·8 square feet.

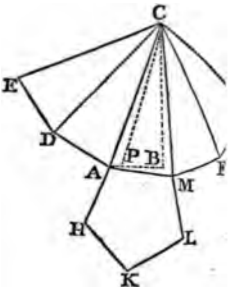
5. What is the whole surface of a hexagonal pyramid, side of the base being 36 inches, and the slant height 20 f

Ans. 203·383

6. What is the whole surface of a rectangular pyramid sides of the base 40 and 30 inches, and the slant height the greater side 20·04, and upon the less side 20·07 feet?

Ans. 125·3083

TO FORM A PYRAMID WITH PASTEBOARD.



Draw AB, and make BC perpendicular to it, and make AB the radius of the circle circumscribing the base, and PB the radius of the inscribed circle. Then if the axis of the pyramid be given, make BC equal to it; or if the slant perpendicular be given, make PC equal to it; or if the slant side be given, make AC equal to it, and from C describe an arc through A, and in it place AD, DE, AM, MF, &c equal to a side of the base, and join CD, CE, CM, & upon AM make the base AHKLM. This figure bei out, and folded along the lines till the sides meet, wil the pyramid, and its area is therefore the surface.

Prob. VI. To find the solid content of a pyra

Rule. Find the area of the base, and multiply it height, and one-third of the product will be the conten

Note. A pyramid is the third part of a prism, wh the same base and altitude. See Appendix, Prop. 81.

1. Required the content of a square pyramid, of which the perpendicular height is 14 feet, and a side of the base 43 inches.

	F.	I.	
	3	7	
	3	7	
	12	10	1
$14 \times \frac{1}{3} =$	4	8	
Ans. Content	59	11	

2. Required the content of a pentagonal pyran height 12 feet, each side of the base 24 inches.

Ans. 27·5276 cu

3. Required the content of a hexagonal pyramid, of which the axis is 9 feet, and each side of the base 29 inches.

Ans. $2{\cdot}5980762 \times 29 \times 29 \times 9 \times \frac{1}{3} \div 144 = 45{\cdot}52046$ cub. feet.

4. Required the content of an octagonal pyramid, the axis 13 feet, each side of the base 35 inches.

Ans. 177·992365 cubic feet.

5. Required the content of a triangular pyramid, the height 22 feet, and each side of the base 39 inches.

Ans. 33 cubic feet 934 inches.

6. Required the content of a triangular pyramid, the perpendicular height 24 feet, and the sides of the base 34, 42, and 50 inches. Ans. 39 cubic feet 406·77 inches.

PROB. VII. To find the surface of a cone.

A CONE is a round solid, which has a circle for its base, and tapers uniformly to a point at the top.

RULE. Multiply half the circumference of the base by the sum of the slant side and the radius of the base: the product is the whole surface. See Appendix, Prop. 79, Ex. 1.

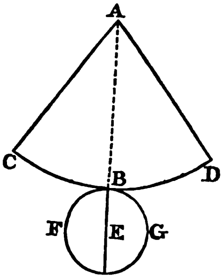
1. Required the surface of a cone, which has 10 feet for its slant side, and 32 inches for the diameter of the base.

$$\begin{array}{r} 3{\cdot}1416 \\ 1\frac{1}{3} \\ \hline 4{\cdot}1888 \\ 11\frac{1}{3} \\ \hline \end{array}$$

Ans. Whole surface 47·4731 square feet.

TO FORM A CONE WITH PASTEBOARD.

Multiply 180° by the radius of the base, and divide it by the slant side to get the angle at the vertex. Draw AB, and make BAC and BAD each equal to the angle at the vertex. Make AB the slant side, and from A describe the arc CBD. Make BE the radius of the base, and from E describe the circle BFG. The figure thus formed is the surface of the cone; and if it be bended till AC meet AD, it will give the form of the cone.



2. Required the surface of a cone, the slant side 14 feet, the circumference of the base 92 inches.

Ans. 58·344 square feet.

3. Required the surface of a cone, the slant side 10 feet, the radius of the base 2 feet 5 inches.

Ans. 94·2698 square feet.

4. Required the surface of a cone, the slant side 18 feet, the diameter of the base 42 inches. Ans. 108 feet $83\frac{3}{4}$ inches.

5. Required the surface of a cone, the slant side 9 feet, the diameter of the base 36 inches. Ans. 49 sq. feet 69 inches.

Prob. VIII. To find the solid content of a cone.

Rule. Multiply the area of the base by the perpendicular height, and one-third of the product will be the content.

Note. The cone is the third of a cylinder, having the same base and altitude. See Appendix, Prop. 80, Cor. 2, and Prop. 81.

1. Required the content of the cone ABC–D, of which the perpendicular height DO is 14 feet, and the diameter AC of the base 43 inches.

$$\begin{array}{rl} & \cdot 7854 \\ 43^2 = & 1849 \\ \hline 144 \times 3 = 432 \,| & 1452\cdot 2046 \text{ square inches.} \\ \hline & 3\cdot 36158 \\ & 14 \\ \hline \end{array}$$

Ans. Content 47·0622 cubic feet.

2. Required the content of a cone, of which the axis is 9 feet, and the circumference of the base 7 feet 10 inches.

Ans. 14·6489 cubic feet.

3. Required the content of a cone, the slant side 15 feet, the radius of the base 19 inches.

The axis is 178·994 inches. Ans. 39·1589 cubic feet.

4. Required the content of a cone, the axis 18 feet, and the diameter of the base 42 inches. Ans. 57 cub. ft. 1256 inches.

5. Required the content of a cone, the diameter of the base 12·7324 feet, and the perpendicular height 107·923 feet.

Ans. 4580 cubic feet $705\frac{1}{2}$ inches.

Prob. IX. To find the surface of a frustum of a pyramid or cone.

A **Frustum** is the portion which remains, after a part has been cut off from the top by a plane parallel to the base.

Rule. Add the perimeters or circumferences of the two bases together, and multiply half the sum by the slant height for the curve surface, to which add the areas of the two bases to get the whole surface.

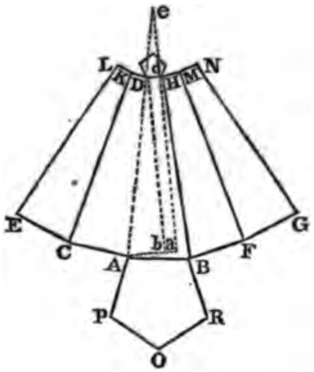
1. Required the surface of the frustum of a square pyramid, the sides of the bases being 40 and 26 inches, and the slant height 10 feet.

	40	
	40	
40		26
26	12 | 1600	26
66	$133\cdot\dot{3}$	12 | 676 bases.
2		
		$56\cdot\dot{3}$
132		$133\cdot\dot{3}$
10		1320·0
1320 curve surface.		12 | $1509\cdot\dot{6}$

Ans. Whole surface $125\cdot80\dot{5}$ square feet.

2. Required the whole surface of a frustum of a pentagonal pyramid, the perpendicular height 11 feet, and the sides of the bases 18 and 34 inches. Ans. 137 square feet 25 inches.

TO FORM A FRUSTUM WITH PASTEBOARD.



Make A*a* and *ab* equal o the radii of the circles escribed about the bases, nd draw *ad* and *b*D perendicular to A*a*, and make ither *b*D the axis, or AD he slant side of the frustum, nd produce *ad* and AD till hey meet in *e*. From *e* escribe circles through A nd D, and in them place raight lines AB, AC, &c. nd DH, DK, &c. equal to e sides of the bases, and in BH, CK, &c.; or if e frustum be that of a ne, make *ae*E, *ae*G, the angle at the vertex. Lastly, upon B, DH, make the bases. Then the figure will be the surce; and if it be folded along the lines, or bended, it will rm the frustum.

3. Required the surface of a frustum of a cone, the diame-

ters of the bases being 43 and 23 inches, and the slant height 9 feet. Ans. 90·7246 square feet.

4. From a cone, of which the circumference of the base is 10 feet, and its slant height 30 feet, a cone has been cut off, of which the slant side is 8 feet. Required the curve surface of the remaining frustum. Ans. $139\frac{1}{3}$ square feet.

5. Required the surface of a frustum of a cone, the perpendicular height of the frustum 13 feet, and the radii of the bases 15 and 24 inches. Ans. 150 square feet 61 inches.

PROB. X. To find the solid content of a frustum of a pyramid or cone.

GENERAL RULE. Find the areas of the two ends, and take the square root of their product: this added to the two areas, and the sum multiplied by a third of the perpendicular height, will give the solid content.

PARTICULAR RULE. If the base be a circle, or a regular polygon, add a diameter, or a side of the greater base, to one of the less, and from the square of the sum subtract the product of these diameters or bases: the remainder, multiplied by the number belonging to the figure, and by a third of the height, will give the content.

NOTE. If $A =$ diameter or side of the greater base, and a that of the less, and h the height of the frustum, and p the proper multiplier, the height of the complete cone or pyramid is $= Ah \div d$ (putting $d = A - a$), and therefore its content is $A^2p \times Ah \div 3d = A^3ph \div 3d$.

In like manner, the part of the cone which is cut off is $a^3ph \div 3d$; and therefore the content of the frustum is $(A^3 - a^3)ph \div 3d = ((A + a)^2 - Aa)\frac{1}{3}ph$. See Appendix, Prop. 81, Cor. 4.

1. Required the content of the frustum of a square pyramid, the sides of the bases being 15 and 6 feet, the height 24 feet.

$$\begin{array}{rr} 15 & \qquad 15 \\ 6 & \qquad 6 \\ \hline 21 & \qquad 90 \\ 21 & \\ \cline{1-1} 441 & \\ 90 & \\ \cline{1-1} 351 & \\ 8 & \\ \cline{1-1} \end{array}$$

Ans. Content 2808 cubic feet.

2. Required the content of the frustum of a triangular pyramid, the height of the frustum 14 feet, the sides of the greater base 21, 15, and 12, and those of the lesser base 14, 10, and 8 feet.

The areas of the bases are $36\sqrt{6}$ and $16\sqrt{6}$, and the square root of their product $24\sqrt{6}$; therefore $(36\sqrt{6}+16\sqrt{6}+24\sqrt{6}) \times \frac{1}{3} \times 14 = 868{\cdot}7523621$ cubic feet the content.

3. Required the content of the frustum of a pentagonal pyramid, the sides of the bases being 42 and 23 inches, and the height 16 feet. Ans. 207·668 cubic feet.

4. Required the content of the frustum of a cone, the diameters of the bases 38 and 27 inches, the height 11 feet.
Ans. 63·9756 cubic feet.

5. Required the content of a mast 57 feet high, and the girths at its ends 63 and 38 inches. Ans. 81·972 cubic feet.

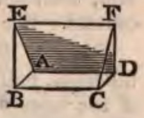
6. Required the content of the frustum of a cone, the height 35 feet, and the bases ellipses, the axes of the greater base 44 and 32, and those of the lesser base 12 and 15 inches.
Ans. 133 cubic feet 141·08 inches.

PROB. XI. To find the superficial and the solid contents of a wedge.

A WEDGE has a rectangle for its base, and its opposite side is a straight line parallel to the base, called its Edge. Its surface consists of a rectangle, two parallelograms or trapeziods, and two triangles, all of which may be easily found.

RULE FOR THE SOLID CONTENT. To twice the length of the base add the length of the edge, and multiply the sum by the breadth of the base, and by one-sixth of the perpendicular from the edge upon the base: the product will be the content. See Appendix, Prop. 82.

1. Required the superficial and the solid contents of a wedge ABCD–EF, of which the sides of the base are BC 36 and BA 9 inches, the edge EF 44 inches, and the perpendicular height 22 inches.



	F.	I.		
	3	0		
	3	0		
	3	8		
	9	8		
	0	9		
	7	3		
	1	10	Perp.	
6	13	3	6	
Ans.	2	2	7	solid content.

2. Required the content of a wedge, of which the height is 25 inches, the edge 28 inches, and the sides of the base 34 and 10 inches. Ans. $2{\cdot}3\dot{1}4\dot{8}$ cubic feet.

3. How many solid feet are in a wedge, of which the base is 40 inches long and 10 inches broad, and each of the ends is inclined to the base in an angle of 70°, the edge being 30 inches? Ans. 1·45748 cubic feet.

4. How many solid feet are in a wedge, of which the sides of the base are 35 and 15, and the length of the edge 55 inches, and the height $17\frac{5}{20}$ inches?

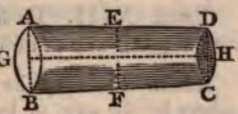
Ans. 3 cubic feet $175\frac{3}{8}$ inches.

PROB. XII. To find the content of any solid, of which the bases are parallel, and the greatest and least thicknesses are at its ends.

RULE. Find the areas of the two bases, and also the area of a section parallel to, and equidistant from, the bases; then to four times the middle area add the other two areas, and the sum, multiplied by one-sixth of the length, will give the solid content. See Appendix, Prop. 82, Cor. 2.

NOTE 1. When the sides of the solid are straight between the bases, half the sum of two corresponding sides or diameters of the bases will give the corresponding side or diameter of the middle section.

NOTE 2. When the extreme thicknesses are not at the ends, divide the solid into portions which have their extreme thicknesses at their ends. Find the contents of these portions separately, and add them: the sum will be the content of the whole.



1. A round solid ABCD, has its length GH 14 feet, and the diameters of the bases AB 94, and CD 21 inches, and the diameter EF of the middle section 27 inches. Required its content.

94	21	54
94	21	54
8836	441	2916
		8836
		441
		12193
		$\cdot7854 \times \frac{1}{6} = \cdot1309$
		1596·0637
		14
		144)22344·8918

Ans. 155·17286 cubic feet the content.

2. Required the content of the prismoid ABCD–EFGH, of which the height is 22 feet, the upper base ABCD is a rectangle, of which the sides are AB 43, and BC 23 inches, and the under base EFGH a square, of which the side EF is 37 inches. Ans. 182·2639 cubic feet.

3. Required the capacity of a waggon $47\frac{1}{4}$ inches deep, and the inside dimensions are, at the top $81\frac{1}{2}$ and 55 inches, and at the bottom 41 and $29\frac{1}{2}$ inches.

Ans. 126340·59375 cubic inches, = 455·6525 imp. gallons.

4. Required the content of a cylindroid 10 feet long, the upper base is an ellipse, of which the axes are 39 and 25 inches, and the under base a circle, of which the diameter is 35 inches. Ans. 60·3594 cubic feet.

5. What is the content of a log of wood, of which the length is 19 feet, and both the bases are rectangles, of which the sides of the lower are 48 and 36 inches, and those of the higher 32 and 21 inches, and the sides of the middle section are 45 and 34 inches? Ans. 187·361 cubic feet.

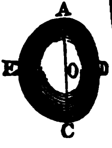
6. What is the content of a round solid, of which the whole length is 37 feet; the greatest girt, 77 inches, is 16 feet from the greater end, of which the girt is 54, and the middle girt 67; also, the girt at the lesser end is 36 inches, and the middle girt 59 inches. Ans. 80 cubic feet $693\frac{1}{2}$ inches.

Prob. XIII. To find the surface of a sphere, or of any segment or zone of it.

Rule. Multiply the circumference of a great circle of the sphere by the axis, or by the part of it corresponding to the segment or the zone required: the product will be the surface. See Appendix, Prop. 83, Cor. 3.

Note. The surface of a sphere, or any part of it, cut off by a plane or planes perpendicular to the axis, is equal to the curve surface of the circumscribing cylinder, which has the same axis, or to the corresponding part of it. Also, the whole surface is four times the area of one of its great circles.

1. Required the surface of a globe AECD, of which the axis AC is 18 inches.



$$\begin{array}{r} 3{\cdot}1416 \\ 18 \\ \hline 56{\cdot}5488 \\ 18 \\ \hline \end{array}$$

Ans. Surface 1017·8784 square inches.

2. Required the surface of a segment of a sphere, the axis 54 inches, and the height of the segment 18 inches.

Ans. 21·2058 square feet.

3. Required the surface of a zone of a sphere, the axis 72 inches, and the height of the zone 24 inches.

Ans. 5428·6848 square inches.

4. Required the surface of the moon, of which the diameter is 2180 miles, supposing her to be a perfect sphere.

Ans. 14930139·84 square miles.

5. Required the surface of the earth, supposing it to be a perfect sphere, of which the axis is 7912 miles; and also the surface of each of its zones, supposing the torrid zone to extend $23\frac{7}{15}°$ on each side of the equator, the frigid zones $23\frac{7}{15}°$ round the poles, and the breadth of each of the temperate zones to be $43\frac{1}{15}°$.

Ans. The part of the axis corresponding to each of the frigid zones is 327·193, to each temperate zone is 2053·46624, and to the torrid zone is 3150·68104; therefore the surface of each frigid zone is 8132807·2, of each temperate zone is 51041534·0, and of the torrid zone is 78314213·42, and the whole surface is 196662895·83 square miles.

Prob. XIV. To find the solid content of a sphere.

Rule. Multiply the cube of the axis by ·5236, which is $\frac{2}{3}$ of ·7854. See Appendix, Prop. 83.

NOTE. A sphere is two-thirds of its circumscribing cylinder.

1. Required the solidity of the sphere, of which the axis is 16 inches.

$$\begin{array}{r} 16 \\ 16 \\ \hline 256 \\ 16 \\ \hline 4096 \\ \cdot 5236 \\ \hline \end{array}$$

Ans. Content 2144·6656 cubic inches.

2. Required the solidity of a sphere, the axis 3 feet 6 inches.
Ans. 22·449 cubic feet.

3. Required the solidity of a sphere, the axis 19 yards.
Ans. 3591·3724 cubic yards.

4. Required the solidity of the moon, supposing her a perfect sphere, the axis 2180 miles.
Ans. 5424617475·2 cubic miles.

5. Required the solidity of the earth, supposing it to be a perfect sphere, and its axis 7912 miles.
Ans. 259332805349·95 cubic miles.

PROB. XV. To find the solid content of a segment of a sphere.

CASE I. When the axis and the height of the segment are given. See Appendix, Prop. 83, Cor. 1.

RULE I. From three times the axis subtract twice the height; multiply the remainder by the square of the height, and by ·5236: the product will be the content.

1. Required the content of a segment 13 inches high, cut off from a sphere, of which the axis is 48 inches.

$(3 \times 48 - 2 \times 16)\, 13^2 \times \cdot 5236 = 10441 \cdot 6312$ cubic inches.

2. Required the content of the frigid zone of the earth, the height 327·2 miles, and the axis 7912.
Ans. 1293879017 cubic miles.

3. Required the content of a segment, of which the height is 57, and the axis 153 inches.
Ans. 339 cubic feet 1113 inches.

4. Required the content of a segment, of which the height is $\frac{3}{8}$ of the axis. Ans. ·16567 cubes of the axis.

CASE II. When the height and the radius of the base of the segment are given. See Appendix, Prop. 83, Cor. 1.

RULE II. To three times the square of the radius add the

square of the height, and multiply the sum by the height, a by ·5236: the product is the content.

5. Required the content of the segment BCD, of which the height CE is 13 inches, and the radius BE of the base 21 inches.

Ans. $(3\times 21^2+13^2)\times 13\times \cdot 5236=$ 10155·7456 cubic inches.

6. Required the content of the segment, of which the height is 3 feet, and the diameter of the base 9 feet. Ans. 109·5633 cubic f

7. Required the content of the segment, of which height is 12, and the radius of the base 48 inches.

Ans. 25 cubic feet 1134 incl

8. Required the content of the segment, of which height is 7 yards, and the diameter of the base 84 yards.

Ans. 19575·8332 cubic yaı

PROB. XVI. To find the solid content of the mid zone of a sphere.

RULE. From the square of the axis, or greatest diame subtract one-third of the square of the height, and multi the remainder by the height, and by ·7854. See Append Prop. 83, Cor. 1.

NOTE. Instead of subtracting one-third of the square of height from that of the axis, add two-thirds of the square the height to the square of the least diameter.

1. Required the content of the middle zone of a sphere, which the axis is 44 inches, and the height of the zone inches.

$(44^2+\frac{1}{3}\times 14^2)14\times\cdot 7854=20569\cdot 1024$ cubic inches.

2. Required the content of the middle zone of a sphere, which the height is 4 feet, and the least diameter 3 feet.

Ans. 61·7848 cubic fe

3. Required the content of the middle zone of a sphere, which the height is 24 inches, and the least diameter 18 inch

Ans. 13345·5168 cubic inch

4. Required the content of the middle zone of a sphere, which the height is 3, and the least diameter 5 yards.

Ans. 73·0422 cubic yar

5. Required the solidity of the torrid zone of the ear the axis being 7912, and the height of the zone 3150·681 miles. Ans. 146717436823 cubic mil

PROB. XVII. To find the solid content of any zo of a sphere.

RULE. Add the squares of the radii of the two ends, and

third of the square of the height, and multiply the sum by twice the height, and by ·7854. See Appendix, Prop. 83, Cor. 1.

1. Required the solid content of a spherical zone, of which the height is 10, and the diameters at its ends 12 and 8 feet.

$(6^2+4^2+\frac{1}{3}\times 10^2)\times 2\times 10\times \cdot 7854=1340\cdot 416$ cub. feet.

2. Required the solid content of a spherical zone, of which the height is 14, and the diameters at its ends 16 and 12 inches.

Ans. 2 cubic feet 179·878 inches.

3. Required the solid content of a spherical zone, of which the height is 9 yards, and the radii at its ends 14 and 10 yards. Ans. 4566 cubic yards $8\frac{1}{2}$ feet.

4. Required the solid content of a spherical zone, of which the height is 11, and the diameters 18 and 13 feet.

Ans. 104 cubic yards $18\frac{1}{2}$ feet.

5. Required the solid content of a spherical zone, of which the height is 23, and the radii 27 and 18 inches.

Ans. 25 cubic feet 1213·8464 inches.

6. The height of the temperate zone of the earth is 2053·46624 miles, and the squares of the greatest and least radii are 13168239 and 2481697 square miles. Required its content. Ans. 55013866469·728 cubic miles.

Prob. XVIII. To find the solid content of a spheroid.

A Spheroid is a solid, generated by the revolution of an ellipse about one of its axes. If it revolve about the greater axis, the solid generated by it is called an Oblong Spheroid; and if it revolve about the lesser axis, the solid is called an Oblate Spheroid.

The axis about which the ellipse revolves is called the Axis, and the other is called the Greatest Diameter, of the spheroid.

Rule. Multiply the square of the greatest diameter by the axis, and by ·5236: the product is the content. See Appendix, Prop. 83, Cor. 2.

1. Required the solid content of an oblong spheroid, the axes of the generating ellipse being 54 and 36 inches.

$$\begin{array}{r} 36 \\ 36 \\ \hline 1296 \\ 54 \\ \hline 69984 \\ \cdot 5236 \\ \hline \end{array}$$

Ans. Content 36643·6224 cubic inches.

NOTE. If a circle be described upon either axis of and
and both revolve about that axis, the spheroid generat
the ellipse will be to the sphere described by the circle, a
circle described by the revolving axis of the ellipse i
circle described by the diameter of the circle; and so i
segment or frustum of the spheroid to the correspondi
ment or frustum of the sphere.

2. Required the content of the oblate spheroid ABCD, the axes of the generating ellipse being 42 and 30 feet.

Ans. 27708·912 cubic feet.

3. Required the content of an oblong, and also of an oblate spheroid, the axes of each ellipse being 48 and 36 inches.

Ans. The oblate 43429·4784, and the oblong 32!
cubic inches.

4. Required the content of an oblong spheroid, of w
axes are 50 and 30 yards. Ans. 23562 cul

5. Required the content of an oblong, and of :
spheroid, the axes of each ellipse being 25 and 15 in
Ans. Oblong 2954¼ cubic inches, oblate 4908¾ cub

PROB. XIX. To find the solid content of a
of a spheroid.

RULE. Find the spherical segment which has
height and the same axis; then, if the base be per
to the fixed axis, the square of that axis is to the sq
other as the spherical to the spheroidal segment. I
revolving axis be perpendicular to the base, that ax
fixed one as the spherical to the spheroidal segm
Appendix, Prop. 83, Cor. 2.

1. The height CG of the segment ECF of the oblong spheroid ABCD, of which the base is perpendicular to the fixed axis, is 16, and the axes are AC 48 and BD 38 feet. Required the content.

$$
\begin{array}{r}
3 \times 48 = 144 \\
2 \times 16 = 32 \\
\hline
112 \\
16^2 = 256 \\
\hline
28672 \\
\cdot 5236 \\
\hline
\end{array}
$$

Ans. $48^2 : 15012{\cdot}6592 :: 38^2 : 9408{\cdot}975$

equired the content of a segment of an oblate spheroid, perpendicular to the fixed axis, the height 12, and the and 30 inches. Ans. 10704·5622 cubic inches.

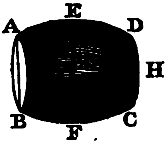
equired the content of a segment of an oblong spheroid, parallel to the fixed axis, the height 14, and the axes 45 inches. Ans. 13177·127 cubic inches.

equired the content of a segment of an oblate spheroid, parallel to the fixed axis, the height 18, and the axes 42 feet. Ans. 16245·3393 cubic feet.

B. XX. To find the solid content of the middle f a spheroid.

E. To twice the area of the greatest base add the area least base, and multiply the sum by one-third of the or height: the product will be the solid content. See lix, Prop. 83, Cor. 1.

equired the content of the middle BCD of an oblong spheroid, the eing perpendicular to the fixed he height GH 48, the greatest r EF 42, and the least AB 32



32	42
32	42
1024	1764
	1764
	1024
	4552
	16
	72832
	·7854

s. Content 57202·2528 cubic inches.

Required the content of the middle zone of an oblong id, the bases parallel to the fixed axis, the height 28, meters of the greatest base 54 and 42, and those of the 5 and 25 inches.

. $(2 \times 54 \times 42 + 35 \times 25) \times \frac{1}{3} \times \cdot 7854 \times 28 =$ ·7944 cubic inches.

Required the content of the middle zone of an oblate id, the bases perpendicular to the fixed axis, the height d the diameters of the greatest and least bases 46 and . Ans. 28233·5592 cubic feet.

4. Required the content of the middle zone of an oblate spheroid, the bases parallel to the fixed axis, the height 12, the diameters of the greatest base 35 and 50, and those of the least base 20 and 28 feet. Ans. 12754·896 cubic feet.

5. Required the content of the middle zone of an oblong spheroid, the bases perpendicular to the fixed axis, the length 40, and the diameters 30 and 18 inches.

Ans. 12 cubic feet $1506\frac{1}{2}$ inches.

6. Required the content of the middle zone of an oblate spheroid, the bases parallel to the fixed axis, the length 40 inches, and the diameters of the greatest base 50 and 30, and of the less 30 and 18 inches. Ans. 21 cubic feet 782·88 inches.

Prob. XXI. To find the solid content of a parabolic conoid.

A Conoid is generated by the revolution of a curve about its axis.

Rule. Multiply the area of the base by half the height: the product will be the content. See Appendix, Prop. 84.

1. Required the content of the parabolic conoid ABC, of which the height BD is 36, and the diameter AC of the base 42 inches.

$$42^2 = 1764$$
$$\times 18$$
$$31752$$
$$\times \cdot 7854$$

Content 24938·0208

2. Required the content of a parabolic conoid, of which the height is 54, and the diameter of the base 40 feet.

Ans. 33929·28 cubic feet.

3. Required the content of a parabolic conoid, of which the height is 16, and the diameter of the base 36 inches.

Ans. 8143·027 cubic inches.

4. Required the content of a parabolic conoid, of which the height is 30 inches, and the diameter of the base 40 inches.

Ans. 10 cubic feet 1569·6 inches.

5. Required the content of a parabolic conoid, of which the height is 27, and its parameter 12 inches.

Ans. 7 cubic feet 1645·3584 inches.

Prob. XXII. To find the solid content of a frustum of a paraboloid.

Rule. Multiply the sum of the squares of the diameters of

the bases by half the height, and by ·7854: the product will be the content. See Appendix, Prop. 84, Cor.

1. Required the content of the frustum EACF (last figure) of a paraboloid, of which the height DG is 12, and the radii of the bases are EG 20, and AD 28 inches.

$$\begin{array}{r} 28^2 = 784 \\ 20^2 = 400 \\ \hline 1184 \\ 6 \\ \hline 7104 \\ 3{\cdot}1416 \\ \hline \end{array}$$

Content 22317·9264 cubic inches.

2. Required the content of the frustum of a paraboloid, of which the height is 38, and the diameters of the bases 32 and 20 feet. Ans. 21249·7824 cubic feet.

3. Required the content of a cask consisting of two frustums of a parabolic conoid joined at their greatest ends, the greatest diameter 34 inches, the least 27, and the whole length 42 inches.

Ans. 31090·059 cubic inches, = 112 imperial gallons 1 pint.

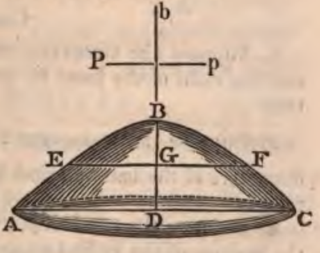
4. Required the content of a cask, the length 40, and the diameters 32 and 26 inches. Ans. 15 cubic feet 783·6 inches.

5. Required the content of a cask, the length 45, and the diameters 40 and 20 inches. Ans. 20 cubic feet 783 inches.

Prob. XXIII. To find the solid content of a hyperbolic conoid.

Rule. Find the content of a cylinder having the same base and altitude with the hyperboloid; then, as the sum of the transverse axis and the height to the sum of this axis and two-thirds of the height, so is half the cylinder to the content of the hyperboloid. See Appendix, Prop. 85, Cor. 1.

1. Suppose the height BD to be 10, the radius of the base AD 12, and the transverse axis Bb 30 inches. Required the content.



$$\begin{array}{r} 3{\cdot}1416 \\ 144 \\ \hline 452{\cdot}3904 \\ 5 \\ \hline \end{array}$$

$40 : 2261{\cdot}952 :: 36\frac{2}{3} : 2073{\cdot}456$ cubic inches.

2. Suppose the height 14, the radius of the base 48, and the transverse 60 feet. Required the content.

Ans. 47472·4629 cubic feet.

3. Suppose the height 22, the radius of the base 60, and the transverse axis 96 feet. Required the content.

Ans. 116675·829 cubic feet.

4. Suppose the height 49, the radius of the base 78, and the transverse 124 inches. Required the content.

Ans. 424069·15 cubic inches.

5. Suppose the height 55, the radius of the base 96, and the transverse 84 inches. Required the content.

Ans. 691191·778 cubic inches.

Prob. XXIV. To find the content of a frustum of a hyperboloid.

Rule. Find a fourth proportional to the transverse, the conjugate, and the altitude, and subtract a third of its square from the sum of the squares of the radii of the bases: the remainder, multiplied by twice the altitude, and by ·7854, will give the content. See Appendix, Prop. 85, Cor. 2.

1. Suppose the transverse B*b* 270, the conjugate P*p* 108, the height DG 10, and the radii of the bases AD 24 and EG 16 inches. Required the content of the frustum.

$$270 : 108 :: 10 : 4 \qquad 4 \times 4 = 16, \quad 16 \div 3 = 5\tfrac{1}{3}$$

$$24^2 = 576, \quad 16^2 = 256, \quad 576 + 256 = 832$$

$$832 - 5\tfrac{1}{3} = 826\tfrac{2}{3}, \quad 826\tfrac{2}{3} \times 20 = 16533\cdot\dot{3}, \quad \times \cdot7854$$

Content 12985·28 cubic inches.

2. Suppose the transverse 200, conjugate 350, height 14, and the radii of the bases 36 and 20 feet. Required the content.

Ans. 32897 cubic feet.

3. Suppose the transverse 270, conjugate $\frac{108}{\sqrt{10}}$, height 40, diameters of the bases 32 and 24 inches. Required the content.

Ans. 24596·6336 cubic inches.

4. Suppose the transverse 30, conjugate 18, height 5, and the squares of the radii 144 and 194·4 inches. Required the content.

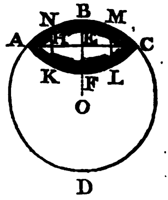
Ans. 2634·2316 cubic inches.

5. Suppose the transverse 45, conjugate 27, height 9, diameters 72 and 544 inches. Required the content.

Ans. 1064111 cubic inches.

OF SPINDLES.

A **Spindle** is a solid generated by the revolution of a segment about its chord. Thus, if ABC be a segment of the circle ABCD, and it revolve about the chord AC, the solid ABCF generated by the revolution is called a Spindle; AC is the length of the spindle, BF = 2BE is its greatest diameter, and EO, the distance of the chord from the centre of the circle, is called the Central Distance.



Prob. XXV. To find the content of a circular spindle.

Rule. Multiply the area of the generating segment by the half of the central distance, and subtract the product from the third of the cube of half the length of the spindle, and four times the remainder, multiplied by 3·1416, will give the content. See Appendix, Prop. 86, Part 2.

1. Required the content of the circular spindle ABCF, of which the length AC is 40, and its greatest diameter BF 30 inches.

$$15)400(26\tfrac{2}{3}$$
$$15$$

Diameter $2|41\tfrac{2}{3})15{\cdot}000({\cdot}360$ ver. sin. — seg. ·25455055

$1736\tfrac{1}{9}$

Radius $20\tfrac{5}{6}$ 441·9280382

15     $2\tfrac{11}{12}$

Cent. dist. $2|5\tfrac{5}{6}$ 1288·9567781

$\tfrac{1}{2}$ cent. dist. $2\tfrac{11}{12}$ $\tfrac{1}{3}\times 20^3 =$ 2666·66666666

1377·70988856

$3{\cdot}1416\times 4 =$ 12·5664

Content 17312·8535428464

2. Required the content of a circular spindle, of which the length is 24, and the greatest diameter 18. Ans. 3739·576.

3. Required the content of a circular spindle, of which the length is 32, and the greatest diameter 24 inches.

Ans. 8864·181 cubic inches.

4. Required the content of a circular spindle, of which the length is 48, and the greatest diameter 18 inches.

Ans. 6770·318 cubic inches.

5. Required the content of a circular spindle, of which the length is 60, and the greatest diameter 12 inches.

Ans. 3660·251 cubic inches.

PROB. XXVI. To find the content of the middle zone of a circular spindle.

RULE. From the square of half the length of the spindle subtract a third of the square of half the length of the zone, and multiply the remainder by half the length of the zone; also find the area of the space which generates the zone, and multiply it by the central distance, and subtract this from the former product; and twice the remainder, multiplied by 3·1416, will give the solid content. See Appendix, Prop. 86, Part 1.

1. The length GH of the middle zone of the spindle ABCF is 40, and its diameters are BF 32 and KN 24 inches. Required its content.

	$4)400(100$		
	4		
Diameter	$104)4\cdot0000(\cdot0384\frac{8}{13}$	ver. sin. — seg.	$\cdot00994038$
		$104^2 =$	10816
Radius	52		
	16		$107\cdot51515$
		$12 \times 40 =$	480
Central dist.	36		
$16 \times 88 =$	1408 sq. of $\frac{1}{2}$ len. spin.	Gen. space	$587\cdot51515$
$\frac{1}{3} \times 400 =$	$133\frac{1}{3}$		36
	$1274\frac{2}{3}$	2d product	$21150\cdot5454$
	20	1st product	$25493\cdot3333$
1st prod.	$25493\cdot\dot{3}$		$4342\cdot7879$
			$6\cdot2832$
		Content	$27286\cdot605125$

2. Required the content of the middle zone of a circular spindle, the length 20, diameters 18 and 8 feet.

Ans. 3657·15 cubic feet.

3. Required the content of the middle zone of a circular spindle, the length 36, and the diameters 24 and 16 inches.

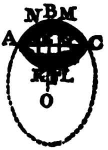
Ans. 13089·676 cubic inches.

4. Required the content of the middle zone of a circular spindle, the length 60, and the diameters 50 and 30 inches.
Ans. 91302·31 cubic inches.

5. Required the content of the middle zone of a circular spindle, the length 80, and the diameters 80 and 40 inches.
Ans. 298353·27 cubic inches.

PROB. XXVII. To find the content of an elliptical spindle.

RULE. Divide three times the area of the generating segment by the length of the spindle, and from the quotient subtract the greatest diameter, and multiply the remainder by four times the central distance, and subtract the product from the square of the greatest diameter; and the remainder, multiplied by the length and by ·5236, will give the content. See Appendix, Prop. 86, Cor. 1, Part 2.



1. Suppose the length AC of the spindle to be 40, the greatest diameter BF 12, the central distance OE 9 inches, and the area of the elliptic segment ABC 167·7345 square inches. Required the content.

Elliptic segment 167·7345
3
40 | 503·2035
12·5801
Greatest diameter 12
·5801
4 × 9 = 36
20·88316
12² = 144
123·1169
40
4924·674
·5236

Ans. Content 2578·55931 cubic inches.

2. Let the length of the spindle be 48, its greatest diameter 18, and the central distance 24. Required the content.
The elliptic segment is 296·8996. Ans. 6800·972.

3. Required the content of an elliptical spindle, the length 60, the greatest diameter 24, and the central distance 32 inches. Ans. 15113·3062 cubic inches.

4. Required the content of an elliptical spindle, the length 36, the greatest diameter 16, and the central distance 20 inches. Ans. 4039·418 cubic inches.

5. Required the content of an elliptical spindle, the length 30, the greatest diameter 14, and the central distance 20 inches. Ans. 2565·432 cubic inches.

PROB. XXVIII. To find the content of the middle zone of an elliptic spindle.

RULE. Find the area of the elliptic segment, of which the chord is equal to the length of the zone, and divide three times this area by its length, and from the quotient subtract the difference between the greatest and least diameters of the zone, and multiply the remainder by eight times the central distance. Subtract the product from the sum of twice the square of the greatest diameter and the square of the least, and the remainder, multiplied by the length and by ·2618, will give the content. See Appendix, Prop. 86, Cor. 1, Part 1.

NOTE. The rules for an elliptical spindle and its zones will give the content of a hyperbolical spindle and of its zones, if the product be added to the squares of the diameters instead of subtracting it.

1. Suppose the length GH of the zone (see last figure) to be 40, its greatest and least diameters FB 32 and KN 24, the central distance OE 4 inches, and the area of the elliptical segment cut off by the straight line KL 109 square inches. Required the content of the zone.

Elliptic segment, 109	$32^2 = 1024$
3	1024
——	$24^2 = 576$
40 | 327	——
——	2624
8·175	5·6
32 — 24 = 8	——
——	2618·4
·175	40
4 × 8 = 32	——
——	104736
Product 5·60	·2618

Ans. Content 27419·8848 cub. inch

2. Suppose the length of the zone to be 60, its greatest and least diameters 40 and 30, and the central distance 20 inches. Required the content of the zone. Ans. 64058·208 cub. inches

3. Suppose the length of the zone to be 48, its diamete

36 and 28, and the central distance 16 inches. Required the content of the zone. Ans. 42264·72 cubic inches.

4. Suppose the length of the zone to be 30, its diameters 20 and 14, and the central distance 12 inches. Required the content of the zone. Ans. 7757·1035 cubic inches.

5. Suppose the length of the zone to be 36, its diameters 30 and 24, and the central distance 18 inches. Required the content of the zone. Ans. 22316·1 cubic inches.

PROB. XXIX. To find the solid content of a parabolic spindle.

RULE. Multiply the square of the greatest diameter by the length and by ·7854, and $\frac{8}{15}$ of the product is the content. Or multiply by ·418879 to get the content. See Appendix, Prop. 87.

1. Suppose the length AC to be 80, and the greatest diameter BD 32 inches. Required the content.

$$\begin{array}{r} 32^2 = 1024 \\ 80 \\ \hline 81920 \\ \frac{8}{15} \times \cdot 7854 = \cdot 418879 \\ \hline \end{array}$$

Ans. Content 34314·56768 cubic inches.

2. Suppose the length to be 64, and the greatest diameter 20 inches. Required the content.
Ans. 10723·328 cubic inches.

3. Suppose the length to be 84, and the greatest diameter 36 inches. Required the content. Ans. 45601 cubic inches.

4. Suppose the length to be 72, and the greatest diameter 42 inches. Required the content. Ans. 53201 cubic inches.

5. Suppose the length to be 108, and the greatest diameter 38 inches. Required the content. Ans. 65325 cubic inches.

PROB. XXX. To find the content of the middle zone of a parabolic spindle.

RULE. To twice the square of the greatest diameter add the square of the least, and from the sum subtract $\frac{4}{10}$ of the square of the difference of these diameters, and multiply the remainder by the length and by ·2618, to get the content. See Appendix, Prop. 87, Cor.

1. Suppose the length FG to be 40, and the greatest diameter BD 32, and the least HK 24 inches.. Required the content.

$$\begin{array}{rr}
32^2 = 1024 & \\
1024 & \\
24^2 = 576 & \\
\hline
2624 & \\
25{\cdot}6 & \\
\hline
2598{\cdot}4 & \\
40 & \\
\hline
103936 & \\
{\cdot}2618 & \\
\hline
\end{array}
\qquad
\begin{array}{r}
32-24=8 \\
8 \\
\hline
64 \\
{\cdot}4 \\
\hline
25{\cdot}6
\end{array}$$

Ans. Content 27210·4448 cubic inches.

2. Suppose the length to be 42, and the diameters 34 and 27 inches. Required the content.

Ans. 33222·10584 cubic inches.

3. Suppose the length to be 48, and the diameters 36 and 30 inches. Required the content. Ans. 43701 cubic inches.

4. Suppose the length to be 44, and the diameters 34 and 28 inches. Required the content. Ans. 35497 cubic inches.

5. Suppose the length to be 38, and the diameters 30 and 24 inches. Required the content. Ans. 23494 cubic inches.

OF UNGULÆ.

An Ungula, or Hoof, is a part of a solid cut off by a plane inclined to the base.

Prob. XXXI. To find the contents of the parts into which a frustum of a rectangular or square pyramid is cut, by a plane passing through one of the sides of the base.

Rule. One of the parts cut off will be a wedge, of which the content may be found by Prob. XI.; and this subtracted from the content of the whole will give the other part.

1. Let the perpendicular height of the frustum of a square pyramid be 287·9649 inches, and the sides of its bases 15 and 6 inches; and let a plane pass through one of the sides of the lesser base, and cut the side of the frustum at the perpendicular height of 119·98536 inches from that base: the length of the section it makes is 9·75 inches. Required the contents of the parts.

$$\begin{array}{lr}
((15+6)^2-15\times6)\times\frac{1}{3}\times287{\cdot}9649 = 33691{\cdot}8933 & \text{frustum.} \\
(12+9{\cdot}75)\times6\times\frac{1}{6}\times119{\cdot}98536 \quad = 2609{\cdot}6816 & \text{wedge.}
\end{array}$$

Ans. Content 31082·2117 remain.

2. Let the height of the frustum of a rectangular pyramid 30 inches, the sides of the greater base 48 and 36, and se of the lesser base 36 and 27; and let a plane pass rough the lesser side of the greater base, and cut the oppo- e at the height of 20 inches; the length of the section it kes with that side is 30 inches. Required the contents of the rts. Ans. Wedge 15120, remainder 24840 cubic inches.

3. Required the contents of the parts of the frustum of a are pyramid, the sides of the bases 30 and 20, a plane ough the greater base passes through the lesser base, the ght 72 inches.

Ans. Wedge 28800, remainder 16800 cubic inches.

4. Required the contents of the parts of the frustum of a tangular pyramid, the sides of the under base 40 and 30, l of the upper base 24 and 18, and the plane passes through greater sides of the two bases, the height 42 inches.

Ans. Wedge 21840, remainder 11088 cubic inches.

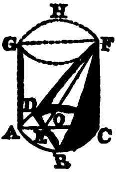
5. Required the contents of the parts of the frustum of a tangular pyramid, the height 60 inches, the sides of the ler base 36 and 28, and of the upper 30 and $23\frac{1}{3}$; a plane es through the greater side of the lower base, and cuts the osite side at the height of 30 inches; the section it makes 3 inches.

Ans. Wedge 14700, remainder 36260 cubic inches.

Prob. XXXII. To find the content of the hoof of cylinder.

Rule. Find the area of the base of the hoof, and multiply by the difference between the radius and the versed sine or ght of the base, and add the product to $\frac{1}{12}$ of the cube of chord of the base, if the height of the base be greater than radius; otherwise subtract them: the sum or difference, ltiplied by the height of the hoof, and divided by the ght of the base, will give the content. See Appendix, p. 88.

Note. If the cutting plane pass through the centre of the e, multiply the square of the diameter by $\frac{1}{6}$ of the height he hoof to get the content.



. Suppose the diameter AC of the base of cylinder to be 50, the height CF of the f 120, and the height or versed sine of its CE 10 inches. Required the content of hoof.

The chord is 40, the segment 279·5595.

$$
\begin{array}{lr}
50)10{\cdot}0({\cdot}2000 \text{ versed sine — Segment} & {\cdot}11182380 \\
\sqrt{40\times 10}=20 & 2500 \\
\qquad 2 & \\
\hline
\text{Chord } 40 & 279{\cdot}5595 \\
12)64000=40^3 & 15 \\
\hline
5333{\cdot}33333 & 4193{\cdot}3925 \\
4193{\cdot}3925 & \\
\hline
1139{\cdot}94083 & \\
12 & \\
\hline
\end{array}
$$

Ans. 13679·29 cubic inches content.

2. Suppose the versed sine of the base to be 40, the rest as before. Required the content.

Here the chord is 40, the base 1683·9359.

Ans. 91777·1154 cubic inches.

3. Suppose the cutting plane to pass through the centre, the rest as before. Required the content.

Ans. 50000 cubic inches.

4. Suppose the diameter of the cylinder 48, the versed sine of the hoof 30, and its height 36 inches. Required the content. Ans. 18604·97 cubic inches.

5. Suppose the diameter of the cylinder 36, the height of the hoof 42, and its versed sine 12 inches. Required the content. Ans. 5167·082 cubic inches.

PROB. XXXIII. To find the content of the hoof of the frustum of a cone.

CASE I. When the cutting plane passes through the extremities of the two bases.

RULE. Take the square root of the product of the diameters at the base and the top of the hoof, and multiply it by the diameter at the top, and take the difference between the product and the square of the diameter of the base, and divide it by the difference of these diameters: the quotient, multiplied by the diameter of the base, by the height, and by ·2618, will give the content. See Appendix, Prop. 90, Case 3.

1. Suppose the diameter of the base of the hoof to be 30, and the diameter of the frustum at the top of the hoof to be 19·2, and the height 18 inches. Required the content.

19·2	$30^2 = 900$
$9{\cdot}2 \times 30 = 24$	460·8
Product 460·8	$30 - 19{\cdot}2 = 10{\cdot}8)\,439{\cdot}2$
	$40{\cdot}\dot{6}$
	30
	1220
	18
	21960
	·2618

Ans. Content 5749·128 cub. inches.

. Suppose the diameter at the base 19·2, and that at the 30, and the height 18 inches. Required the content.
Ans. 2943·55 cubic inches.

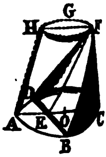
. Suppose the diameter of the base 24, and the diameter he top 18, and the height 36 inches. Required the con- t.
Ans. 7610·6 cubic inches.

. Suppose the diameter of the base 20, that at the top 28, the height 14 inches. Required the content.
Ans. 2406·215 cubic inches.

5. Suppose the diameter of the base 15, that at the top 12, the height 16 inches. Required the content.
Ans. 1340·481 cubic inches.

Case II. When the plane cuts off a part of the base.

Rule. Find the tabular area answering to the quotient of e height of the base by its diameter, and multiply it by the be of that diameter for the first content. Also, from the ight of the base subtract the difference between the dia- eters at the top and the base of the hoof, and take the bular area answering to the quotient of the remainder vided by the diameter at the top, and multiply it by the be of the diameter at the top, and by the quotient of the ight of the base divided by the said remainder, and by the uare root of the same quotient, for another content. Mul- ly the difference of these contents by one-third of the ght of the hoof, and the product, divided by the difference the diameters, will give the content of the under hoof; and s hoof, subtracted from the content of the frustum, will e the other hoof. See Appendix, Prop. 90, Case 2.



1. Suppose the height of the hoof to be 18, the diameter AC of the lower base 30, and the diameter FH at the top 19·2, and that the plane cuts off CE 20 inches height from the lower base. Required the content.

The tabular area of $\frac{20}{30}$ is ·55622573, which, multiplied by 27000, gives 15018·09471 the first content; and the tabular area for $9·2 \div 19·2 = ·4791\frac{2}{3}$ is ·37187178, which, multiplied by $19·2^3$, and by $20 \div 9·2$, and $\sqrt{20 \div 9·2}$, gives 8436·45824, which, subtracted from the former content, leaves 6581·63647; and this, multiplied by 6, and divided by 10·8, gives 3656·4647 the content.

2. Suppose the plane to cut 15 inches for the height of the base, the rest as before. Required the content.

Ans. 2517·8608 cubic inches.

3. Suppose the height of the base 10·8 inches, the rest as before. Required the content. Ans. 1606·41834 cubic inches.

NOTE. In this example, when the height of the base is equal to the difference of the diameters at the base and top, the tabular versed sine for the second area is nothing. Therefore, multiply the first tabular area by the cube of the diameter at the base, and divide the product by the height of the base, for the first content. Also, multiply the height of the base by the diameter at the top, and multiply the square root of the product by the same diameter, and to the product add one-third of itself, for the second content. The difference of these contents, multiplied by one-third of the height of the hoof, gives its content.

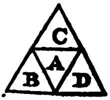
4. Suppose the diameter of the base 36, and at the top 27, the height 24, and the versed sine 18 inches. Required the content. Ans. Hoof 4945·152 cubic inches.

5. Suppose the diameter of the base 24, and at the top 32, the height 42, and the versed sine 16 inches. Required the content. Ans. Hoof 11447·96 cubic inches.

PROB. XXXIV. To form the five regular bodies with pasteboard.

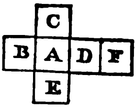
A REGULAR BODY is a solid bounded by similar and regular plane figures. Of these there can be only five.

1. The TETRAEDRON, bounded by four equilateral triangles.



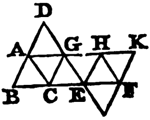
Make the equilateral triangle A, and upon each side of it make an equilateral triangle. The figure, cut out of the paper, and folded at its lines, will form the tetraedron.

2. The HEXAEDRON, bounded by six squares.



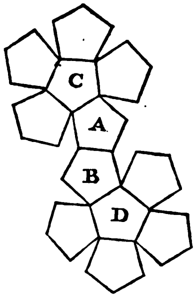
Make the square A, and upon its sides the squares B, C, D, E, and on the outermost side of D make the square F. The figure, cut out and folded, will form the hexaedron.

3. The OCTAEDRON, bounded by eight equilateral triangles.



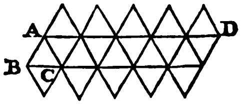
Make the equilateral triangle ABC, and through A draw AK parallel to BC, and make CE, EF, AD, AG, GH, and HK, each equal to BC, and join the points as in the figure. When folded, this figure will form the octaedron.

4. The DODECAEDRON, bounded by twelve pentagons.



Make two regular pentagons A and B on the same straight line, and on the most distant sides of these make the pentagons C and D; then make a pentagon on each of the sides of C and D; and the figure, when folded, will form the dodecaedron.

5. The ICOSAEDRON, bounded by twenty equilateral triangles.



Make the equilateral triangle ABC, and through A draw AD parallel to BC, and lay BC five times on each of the parallels, and join the points as in the figure. This figure, when folded, will form the icosaedron.

Their surfaces are got by finding the area of one of their faces, and multiplying it by the number of them. Thus the square of a linear side, multiplied by ·4330127, and by 4, or by 8, or by 20, will give the surface of the tetraedron, or the octaedron, or of the *icosaedron*; the square of the side, multi-

plied by 6, will give the hexaedron; and the square of the side, multiplied by 1·7204774, and by 12, will give the surface of the dodecaedron.

Or the surface and the solidity of any of the regular bodies may be found from the following Table.

TABLE
OF THE SURFACES AND SOLIDITIES OF REGULAR BODIES.

No. of sides.	Name.	Surface.	Solidity.
4	Tetraedron, . .	1·7320508	0·1178511
6	Hexaedron, . .	6·0000000	1·0000000
8	Octaedron, . .	3·4641016	0·4714045
12	Dodecaedron, .	20·6457788	7·6631189
20	Icosaedron, . .	8·6602540	2·1816950

Rule I. To find the Surface. Multiply the square of the linear side by the proper number in the table under *Surface:* the product will be the surface.

Rule II. To find the Solidity. Multiply the cube of the linear side by the proper number under *Solidity:* the product will be the solid content.

1. Required the surface and the solidity of an octaedron, of which the side is 16 inches.

Ans. $16 \times 16 \times 3{\cdot}4641016 = 886{\cdot}81$ square inches surface.
$16^3 \times {\cdot}4714045 = 1930{\cdot}8728$ cubic inches solidity.

2. Required the surface and the solidity of a dodecaedron, of which the side is 12 feet.

Ans. Surface 2972·992 sq. feet, solidity 13241·8695 cub. feet.

3. Required the surface and the solidity of a tetraedron, of which the side is 2 feet.

Ans. Surface 6·9282 square feet, solidity 0·9428 cubic feet.

4. Required the surface and the solidity of a hexaedron, of which the side is 27 inches.

Ans. Surface 4374 sq. inches, solidity 19683 cub. inches.

5. Required the surface and the solidity of an icosaedron, of which the side is 15 inches.

Ans. Surface 1948·557 sq. inches, solidity 7363·22 cub. in.

Prob. XXXV. To find the convex surface of a solid ring.

Rule. To the thickness of the ring add the inner diameter, to get the axis; multiply this by the thickness, and by 9·8696, to get the surface.

1. Suppose the thickness of the ring 3 inches, and the inner diameter 12 inches. Required its surface.

Ans. $(12+3)\times 3\times 9{\cdot}8696 = 444{\cdot}132$ square inches.

2. Suppose the thickness 2, and the inner diameter 18 inches. Required the surface. Ans. 394·784 square inches.

3. Suppose the thickness 3, and the inner diameter 14 inches. Required the surface. Ans. 503·3496 square inches.

4. Suppose the thickness 5, and the inner diameter 18 inches. Required the surface. Ans. 1135 square inches.

5. Suppose the thickness 6, and the inner diameter 24 inches. Required the surface. Ans. 1776·528 square inches.

Prob. XXXVI. To find the solidity of a ring.

Rule. Multiply the axis by 3·1416 to get the length, and then by the square of the thickness, and by ·7854: the product is the content.

Or multiply the axis by the square of the thickness, and by 2·4674.

1. Required the solidity of a ring 2 inches thick, of which the inner diameter is 18 inches.

$18+2=20$ axis, $20\times 3{\cdot}1416 = 62{\cdot}832$ length.

Ans. $62{\cdot}832\times 4\times {\cdot}7854 = 197{\cdot}393$ cubic inches.

2. Required the solidity of a ring, the thickness 3, and the inner diameter 8 inches. Ans. 244·2726 cubic inches.

3. Required the solidity of a ring, the thickness 4, and the inner diameter 16 inches. Ans. 789·572 cubic inches.

4. Required the solidity of a ring, the thickness 5, and the inner diameter 12 inches. Ans. 1048·65 cubic inches.

5. Required the solidity of a ring, the thickness 6, and the inner diameter 18 inches. Ans. 2131·8445 cubic inches.

SURVEYING.

SURVEYING is the method of determining the magnitude, position, and shape of lines, fields, &c. on the surface of the earth.

For this purpose, various instruments are used for measuring lines and angles.

OF LINES.

Straight lines are measured by applying to them a line of known length, as a foot-rule, a yard, a measuring-line, or a chain.

The CHAIN consists of 100 links, and is distinguished at the end of every 10 links by a brass ring or point. It is made of such a length, that 10 chains in length, and 1 in breadth, contain an acre.

If the length of a pendulum vibrating seconds at Greenwich Observatory be taken 23 times, and the amount be divided into 25 equal parts, each of these parts will be nearly an English yard, or 22 of them an English chain; therefore a link of it will be 7·92 inches.*

The Scotch chain is 74·1196 English feet long, and each link of it is 8·89 inches. The Scotch ell is 37 Scotch inches, = 37·0598 English inches; and 6 ells make a fall.

OF THE OFFSET-STAFF. This is a pole of 10 links length. It is divided into 10 parts, and the last of them subdivided into 10 smaller parts. Its use is for examining the chain, which is liable to stretch with long usage or the roughness of the ground. It is also used for measuring short distances, such as perpendiculars from the principal straight line to the hedges.

OF THE CROSS. This consists of two pair of sights fixed on a pole, at right angles to one another. Its use is to deter-

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* The length of the pendulum vibrating seconds in a vacuum at the level of the sea in the latitude of London, is 39·1393 imperial inches: 23 times the length of this pendulum comes to 900·204 inches, and 25 yards to 900 inches; so that 23 times the length of the pendulum is very nearly 25 yards.

mine the point in which a perpendicular from a corner would meet the principal line that is measured. Move the cross along the line, keeping its extremities in view through one pair of the sights, till the corner from which the perpendicular comes is seen through the other pair of sights: the cross is then at the foot of the perpendicular.

The PERAMBULATOR is sometimes used for measuring roads, &c. It turns upon a wheel, of which the circumference is 8·25 feet, so that 8 revolutions make an English chain in length. The distance measured is pointed out by an index moved by clock-work.

PROB. I. To measure a straight line in the field.

Erect poles at the extremities, and at convenient distances along the line, for showing the direction. Ten arrows of iron or wood are used for pointing the spot to which the chain extends, and for preserving the number of chains. Let the leader, or the person going before, take the end of the chain and the ten arrows; and having stretched the chain, and taken notice that none of the links are involved in one another, let the follower, placing the end of the chain at the extremity of the line, direct him, by waving his hand towards the right or left, into the proper direction. And the leader having fixed an arrow at the end of the chain, let them both go forward with the chain, till the follower comes to the arrow: there let him direct the leader as before, who fixes another arrow, while the follower takes up the former one. Let them proceed thus, till all the arrows are in the hand of the follower, and the chain stretched beyond the last of them; then let the arrows be conveyed to the leader, and let him fix one of them, and proceed in the same way till all the arrows are again changed, or till he has arrived at the end of the line to be measured. And at the last, let the follower reckon the number of changes, the number of arrows in his hand, and the number of links between him and the extremity of the line. Thus, 3 changes 7 arrows and 45 links, make the length of the line to be 3745 links.

The surveyor, while measuring a straight line, ought carefully to take notice of every surrounding object of which the position can be more easily determined from it than from any other line which he intends to measure. He ought to mark the distance at which the line meets a corner, or crosses a boundary, or begins or ceases to run along a hedge, a wall, or a road. He must likewise mark the distances at which perpendiculars or offsets are to be raised; and, in general, every

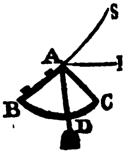
thing which may tend to shorten his other operations in the survey, or will assist him in drawing his plans. When he has settled, by the cross, the place of an offset or short perpendicular, it will be easiest to measure the length of it as he goes along, to save the time and trouble of returning to the place a second time.

It is proper to remark, that the plan ought to be drawn upon paper, with horizontal distances only; otherwise it will be impossible to join several fields together without distortion. For when several lines are to be joined together, a small error in the lengths of some of them will alter the position of others; a circumstance which has a greater tendency to distort the plan, than even the lengths of the lines themselves. It is, however, impossible for a surveyor to ascertain the exact level of every elevation and depression of his lines; but it would be of great advantage to him to take a level at that part which he judges to have a mean inclination. This may be done with the offset-staff thus:—Having laid the chain along that part, place one end of the offset-staff at the uppermost of 10 links on it, and let the assistant take the other end, and a line and plummet hung exactly over the other end of the 10 links on the chain, and let the surveyor apply a pocket or other level to the staff; and when it is level, the line of the plummet will point out on the staff the horizontal length of the 10 links of the chain. Consequently, by using a diagonal scale of 10 to a link, it will point out how much the line is to be diminished to get the horizontal length of it.

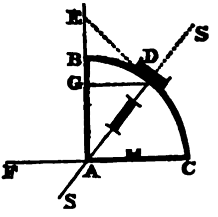
OF THE INSTRUMENTS USED FOR TAKING ANGLES.

Angles in the field are taken either in a vertical or in a horizontal plane. The former are measured by a Quadrant, and the latter by a Theodolite or Circle.

A QUADRANT is the fourth part of a circle of any convenient radius. It is made of brass or wood, and the arc is divided into 90 degrees, and each degree is subdivided into smaller parts. The degrees are numbered from one extremity, called the beginning of the arc, to the other extremity or end of it.



The most simple quadrant, ABC, has a line with a plummet suspended from its centre, as AD, which, when hanging freely, is always perpendicular to the horizon; and *sights*, or a telescope, is affixed to the radius AB, which passes through the 90th degree, or end of the arc, to direct the eye in a straight line towards the object.



Sometimes an index AD, with scopic sights, is made to revolve nd the centre A; in which case a rit-level is fixed to the radius AC, ich passes through the beginning the arc. The telescope is placed ng AD. But sometimes the de- es are numbered from B, and a escope is fixed at D, perpendicular the index AD.

The THEODOLITE is the most complete instrument for sur- ing. It consists of a brass circular plate, the circumference which is divided into 360 degrees, or twice 180 degrees, each degree is subdivided into smaller parts. An index a compass on it is fixed to the centre, and revolves round and on it is erected a semicircle, perpendicular to the e of the instrument, furnished with a telescope perpendi- r to the index of it, which moves round its centre. The of the circle is for taking horizontal angles, and that of semicircle is for taking vertical ones. The instrument is ished with two spirit-levels for placing the plate, and the cope, when at the top of the semicircle, in a horizontal tion; in subservience to which, the tripod upon which instrument stands has four screws, &c. A more particular ription of this instrument, in its most improved state, ld scarcely be intelligible to a learner, without seeing and g it; and it is therefore omitted here.

The CIRCUMFERENTER is a circle, on the centre of which large compass; and the circumference is divided, not only o points and quarters, but also into degrees and parts of a gree. An index or two is moveable about the centre. Its is the same with that of the theodolite; only, when using greater reliance is placed upon the compass. It is chiefly ed for surveying mines.

Large LEVELS, with telescopic sights, are often requisite r finding the elevation of one place above another in feet, &c. nd the surveyor ought also to be possessed of several pocket- els, to be applied when occasion requires them.

Each of the indices of these instruments has a NONIUS, for abling the artist to read off minutes. The nonius is a scale which the number of divisions is greater by one than the mber in the same space upon the arc. If the nonius upy the space of 29 divisions on the arc, it is divided into equal parts, by which means each division will exceed one the nonius by $\frac{1}{30}$ of a division on the arc; so that, by ving forward the index $\frac{1}{30}$ of a division of the arc, the

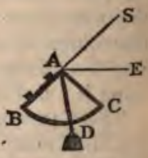
first one on the nonius will coincide with one on the arc; and by moving another $\frac{1}{30}$, the second will coincide, and so on. Consequently, if the arc be divided into half degrees, the nonius will point out minutes.

INSTRUMENTS USED IN DRAWING PLANS.

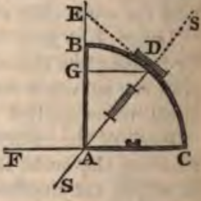
The surveyor ought to be provided with compasses of various sizes, some of which must have very fine points, both of steel and for ink. He ought also to have drawing-pens of different finenesses, for drawing coarse and fine lines; and a number of scales of various sizes, from one chain in an inch to 8 or 10 chains in an inch, which ought to have the divisions marked on the edges for laying down distances without compasses. He will also stand in need of lines of chords, and protractors of different radii; and, for the sake of expedition, he ought to use parallel and perpendicular rulers and reducing scales.

Prob. II. To take a vertical angle in the field.

Vertical angles are denominated Angles of Elevation when the object is higher than the eye, and Angles of Depression when it is lower.

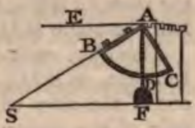


1. *To take an angle of elevation.* If the quadrant ABC have a plummet, place the eye to the limb B, and look through the sights in AB to the object S, and the line and plummet AD hanging freely, will cut off the arc CD from the end C, farthest from the sights, the degrees, &c. of which will be the measure of the angle EAS, contained by the horizontal line AE, and the visual ray AS; for DAE and CAS are right angles.



If the quadrant have a telescope fixed on the index AD, which moves about the centre A: Having levelled the radius AC, and directed the quadrant towards the object S, move the index AD till S is seen at the crossing of the wires of the telescope; then the arc CD is the measure of the angle CAD.

If the telescope be at D, perpendicular to AD, move the index, till, looking through the telescope, the object E is in the centre of the telescope; then the arc BD is the measure of the angle of elevation.



2. *To take an angle of depression.* If the quadrant ABC have a plummet, place the eye at the centre A, and look through the sights in the radius AB to the object S below, and the line of the plummet AD will cut off the arc CD, the measure of the angle of depression EAS; for EAD and BAC are right angles.

If the telescope be on the index AD, place the eye at the limb D, and look down to S through the telescope; and the arc CD is the measure of the angle of depression.

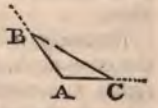
If the telescope be perpendicular to the index, depress the object-glass till the object be seen; and the arc BD between the index and the vertex is the measure of the angle of depression.

PROB. III. To measure a horizontal angle in the field.

First, with the Theodolite. Having placed the instrument at the angular point, and the cipher of the index at the beginning of the degrees on the circle, turn the whole instrument about till a distant pole in one of the sides of the angle be seen in the centre of the telescope; there fix the instrument, and turn the index upon it, till a pole fixed in the other side of the angle be seen in the centre of the telescope: then the degrees, &c. moved over by the index is the measure of the angle.

Secondly, with the Circumferenter or the Compass. Having fixed the instrument, so that the north point of the compass point to the fleur-de-lis, direct the sights to a mark in one side of the angle, and mark the degrees, &c. pointed out by the needle. Then turn the sights towards a mark in the other side of the angle, and again mark the degrees cut by the needle. Their sum or difference, according as they are on different or on the same side of the north or south points, will give the quantity of the angle.

NOTE. The degrees marked show the bearing of the sides of the angle, allowance being made for the variation.

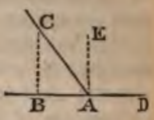


Thirdly, with the Chain. Extend the chain along one of the sides, from the angular point A to B, and along the other side from A to C, and measure from C to B. Then, having drawn the triangle ABC upon paper, the angle BAC may be measured with a protractor, or with the line of chords.

NOTE. If a table of natural sines be at hand, look among

the sines for $\frac{1}{2}$BC, and the degrees, &c. answering to it will be half the angle BAC.

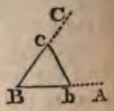
Fourthly, with the Cross. If the angle be acute, as BAC, place the cross at B in one of the sides of the angle, so that one pair of the sights may be directed along AB; and, looking through the other pair of sights, let an assistant mark the point C of the line AC, which is seen through them; and then the angle BAC is determined by measuring AB and BC. If the angle be obtuse, as CAD, it may be determined by measuring its supplement BAC, or by placing the cross at A, so that AD may be seen through one pair of the sights; then let an assistant place a distant mark at E, seen through the other pair of sights; after which measure the angle EAC as before, and add a right angle to it.



Prob. IV. To make or lay down an angle in the field.

First, with the Theodolite. Having placed the instrument at the point at which the angle is to be made, and fixed the index at the beginning of the degrees, turn the theodolite until a mark is seen in the given line, there fix it, and turn the index upon it the proper way over the given number of degrees; then, looking through the telescope, direct an assistant to place a mark.

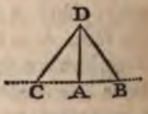
Secondly, with the Chain. The angle must first be made on paper, as ABC. Make B*b* and B*c* each 30, and measure *bc*. Lay 30 links on the given line on the ground from B to *b*; and having reckoned as many links of the chain as are in the sum of B*c* and *cb*, fix the ends of them at B and *b*, and, taking 30 links from B in your hand, go backward till both ends of the chain are equally stretched, and there fix a pin in the ground, which will give *c*.



Prob. V. To raise a perpendicular in the field.

First, with the Theodolite, Circumferenter, &c. At the given point in the line make an angle of 90°, by the last problem.

Secondly, with the Cross. Having placed the cross at A, and directed one pair of the sights to a mark B in the given line, look through the other pair of sights, and cause a mark D to be placed in that direction.



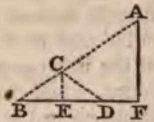
Thirdly, with the Chain. Measure in the given line 30

inks from A to B, and as many from A to C; and, fixing he ends of the chain at B and C, take hold of the 50th link, nd go backwards till both ends of the chain are equally tretched, and there fix a pin at D; then AD will be perpen- icular to BC.

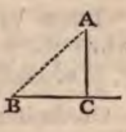
PROB. VI. To drop a perpendicular in the field.

First, with the Cross. Move the cross along the given line, that its extremities appear through one pair of the sights, ntil the given point is seen through the other pair. The strument is then in the point of the line upon which the erpendicular falls.

Secondly, with the Chain. Measure a traight line from the point A to any point 3 of the given line. Let BC be a chain in hat direction. Fix one end of the chain at ;, and with the other go along the given line ill the chain is again stretched, and there nake a mark, as at D. Measure BD, and multiply $\frac{1}{2}$BD by 3A, and cut off two figures from the right of the product: the est will give BF, the distance of B from the foot of the per- endicular AF.

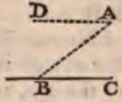


Thirdly, with the Theodolite. Fix the in- trument at any point B of the given line BC, nd measure the angle ABC by Prob. III.; hen fix the instrument at A, and by Prob. IV. make the angle BAC the complement of ABC, nd AC will be the perpendicular required.



PROB. VII. To run a line in the field parallel to a given straight line BC.

Take any point B in the given line BC, and measure the angle ABC contained by BC, and the line directed to the given point A; then at A make the angle BAD equal to ABC, and AD will be the direction of the parallel.



OF THE PLANE-TABLE.

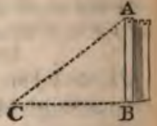
This is an instrument much used in surveying, when the survey is not large, because it gives the plan of the ground, as well as its quantity. It is a rectangular board fixed upon a tripod, with a ball and socket for giving it any inclination. It has a loose frame fitted to it, one side of which is divided into equal parts all around; and the other side is divided into 360 degrees, by lines directed to the centre of the table; and

a compass is fastened to one of the sides of the table. There is a loose index to be used with it, having a telescope placed parallel to its fiducial side; and there are several plane scales upon the index, for laying down the measured distances. A sheet of paper, moistened equally with a sponge, is spread upon the table, and the frame pressed down upon it to keep it fixed. The paper will become smooth when it is dry, and it will then be fit for drawing the plan upon.

An angle may be measured with the plane-table, by placing that side of the frame uppermost which has degrees on it, and proceeding as with the theodolite. Or the angle may be drawn on the table, by directing the index to marks in the sides of the angle in the field; and, in like manner, a given angle may be formed in the field by the table. Also, a perpendicular may be drawn in the field with it, by placing the centre of the instrument at the given point, and turning it, till the index, while cutting the same divisions on opposite sides of the frame, is in the direction of the given line: then, if the index be made to cut similar divisions on the other sides of the table, it will give the direction of the perpendicular.

OF HEIGHTS AND DISTANCES.

PROB. VIII. To find the height of an object A, when the point B on the level ground, directly below it, is accessible.



On the level ground measure any distance BC in a straight line, and at C take the angle of elevation ACB with a quadrant.

1. Suppose BC 236 feet, and ACB 35° 48′. In the triangle ABC, right-angled at B, are given BC 236, and ACB 35° 48′. To find AB. Ans. R : tan. C : : CB : BA 170·208 feet.

NOTE. The height thus obtained is that above the level of the eye of the observer, and must be increased by the height of the eye, to have its height above the level ground. The same is to be done in all the observations on heights.

2. From the bottom of a steeple I measured upon a level plane a straight line 136 feet, and at its extremity I took the elevation of the top of the steeple 47° 25′. Required the height of the steeple. Ans. 147·98 feet.

3. The elevation of a wall, taken from the edge of the ditch 18 feet wide, was 62° 40′. Required the height of the wall, and the length of a ladder to reach the top of it.

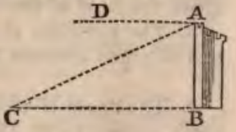
Ans. Height 34·824, ladder 39·2014 feet.

4. At 85 feet from the bottom of a tower, the angle of its elevation was 52° 30′. Required its altitude.

Ans. 110·744 feet.

5. Near the bottom of a hill I took the elevation of its top 54° 40′, and the altitude of the hill was 1156 feet. Required the distance of my station from its top. Ans. 1417·01 feet.

Prob. IX. From the top of a known height AB, to find the distance of an object C, on the plane below.



Take the angle of depression CAD; then, in the triangle ABC, right-angled at B, are given AB, suppose 83 feet, and the angle ACB = DAC, suppose 23° 37′. To find AC or BC.

Ans. Sin. C : R :: AB : AC 207·181 feet, and tan. C : R :: AB : BC 189·829.

Note. If AC be given, AB and BC may be found from it.

2. Let the sloping side of a hill AC be 268 feet, and the angle of depression at its top DAC, be 33° 45′. Required the base BC, and its perpendicular height AB.

Ans. BC 222·834, AB 148·893 feet.

3. From the top of a mast 80 feet high, the angle of depression of another ship's hull was 20°. Required their distance. Ans. 219·798 feet.

4. From the top of a tower 120 feet high I took the depression of two trees 57° and 25° 30′. Required their distances from the tower and from each other.

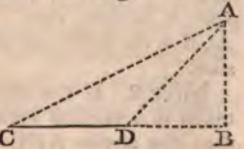
Ans. 77·93 feet, and 251·58 feet, and 173·65 feet.

5. Suppose the mean semidiameter of the sun subtends at the earth an angle of 16′ 7½″; what is his distance from the earth? Ans. 213·2379 semidiameters.

6. From the top of a lighthouse 110 feet high I observed two ships in a straight line from it, and took the angles of depression of their hulls 56° 44′ and 18° 26′. Required their distance from the lighthouse.

Ans. 72·165 feet, and 330·032 feet.

Prob. X. To measure an inaccessible height AB.



On the level ground measure any distance CD, in a straight line towards the height, and at C and D take the angles of elevation ACB and ADB; their difference is CAD. Let CD be 248 yards, ACB 23° 30′, ADB 37° 24′; then CAD is 13° 54′.

Ans. Sin. CAD : sin. ACD :: CD : DA, and R : sin. D DA : AB 250·026 yards; that is, sin. C × sin. D × CD · sin. (D — C) = AB.

Or the difference of the natural cotangents of C and D is the radius as CD to AB.

2. Sailing in a boat, a hill was observed, and the elevation of its top above the level of the sea was 27° 38′. After sailing 540 fathoms, each 5 feet, directly towards the hill, the elevation of its top was 35° 28′. Required the height of the hill above the level of the sea. Ans. 1066·26 fathoms.

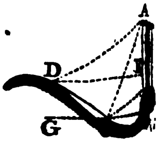
3. The elevation of a hill at the bottom of it was 46°, and at 100 yards distance 31°. Required the height of it.

Ans. 143·145 yards.

4. The angle of elevation of a tower was 26° 30′, and, 75 yards nearer to it, the elevation was 51° 26′. Required the height and distance. Ans. Height 61·97, dist. 49·294 yards.

5. Measured 149 yards towards a hill, and at the extremities of the line the elevations of its top were 29° 17′ and 39° 25′. Required its height. Ans. 263·02 yards.

Prob. XI. To measure a height which has no level ground before it.



Take two stations C and D, in a vertical plane, and measure CD, and at C take the elevation of D above C, viz. GCD 31° 26′, and the elevations or depressions of the top and bottom of the height, viz. ACF 53° 26′, and BCF 18° 32′, and at D take the elevation of the top ADE 22° 30′, and let CD be 286 feet. Since EDC = DCG, ADC = ADE + DCG = 53° 56′, and DAC = ACF — ADE. The triangle ADC has two angles; and the side CD given, to find AC. Then in the triangle ACB are given ACB = ACF ± BCF, and B = 90° ± BCF, and AC; to find AB.

Ans. Sin. DAC : sin. ADC :: DC : CA, and sin. B : sin. ACB :: CA : AB 271·39.

Note 1. If DE be above A, the angle DAC is the sum of ACF and ADE; otherwise it is their difference. Also, in this case ADC is the difference of DCG and ADE; otherwise it is their sum. Also, when F is below B, the angle ACB is the difference of ACF and BCF; otherwise it is the sum.

Note 2. If the stations C and D cannot be conveniently taken in a vertical plane, they may be taken anywhere, and then the angles ADC and ACD must be measured with a sextant, and the triangle ACD will give the side AC.

. At a considerable distance from a hill, I took the eleva- of the top of a tower built upon it, 33° 45′; and mea- ng on level ground 300 feet directly towards the hill, I in took the elevations of the top and the bottom of the er 51° and 40°. Required the height of the tower.

Ans. 46·666 yards.

. At a window on a level with the base of a steeple, I took elevation of its top 40°; and at another window of the e house 18 feet higher, I took again the elevation of the of the steeple 37° 30′. Required the height of the steeple.

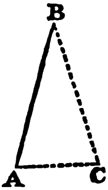
Ans. 210·44 feet.

. The elevation of the top of a hill at one station was 38° Another station was taken 450 feet from the first, but her on a level with it nor in the direction of the hill. At first station, the line from the other station to the top of hill subtended an angle of 67° 30′; and at the second, the from the first to the top of the hill subtended an angle of 8′. Required the height of the hill. Ans. 441·25 feet.

I measured directly up a hill 132 yards: there I took depression of the hill 42°, that of the bottom of a distant e 27°, and that of its top 19°. Required the height of the e.

Ans. 28·637 yards.

ROB. XII. To find the distance of a place A, from accessible object B.



rst. Let B be visible from A. Choose a on C, from which both A and B can be seen. ure AC 650 yards, and take the angles C 72° 22′, and ACB 78° 37′, with the theo- te. Then ABC is 29° 1′, and sin. B : sin. C A : AB 1313·67 yards.

econdly. Let B be not visible from A. ose a station C from which both A and B be seen, and their distances from it mea- ed. Take the angle ACB 75° 38′, and measure AC 358, CB 560 feet.

Ans. (BC+CA)918 : (BC—CA)202 : : tan. ½(A+B) 11′ : tan. ½(A—B) 15° 49·7′; whence BAC is 68° 0·7′, sin. A : sin. C : : CB : BA 585·041.

. A straight line was measured along the bank of a river feet, and at its extremities the angles contained by it, and ight lines directed to a tree upon the opposite bank, were 40′ and 73° 26′. Required the breadth of the river.

ns. 648·366 perp. breadth, and 676·444 feet to the nearest on.

. Straight lines from a station to two places measured 694

Ans. Sin. CAD : sin. ACD : : CD : DA, and R : sin. D : : DA : AB 250·026 yards; that is, $\sin. C \times \sin. D \times CD \div \sin. (D - C) = AB$.

Or the difference of the natural cotangents of C and D is to the radius as CD to AB.

2. Sailing in a boat, a hill was observed, and the elevation of its top above the level of the sea was 27° 38′. After sailing 540 fathoms, each 5 feet, directly towards the hill, the elevation of its top was 35° 28′. Required the height of the hill above the level of the sea. Ans. 1066·26 fathoms.

3. The elevation of a hill at the bottom of it was 46°, and at 100 yards distance 31°. Required the height of it.

Ans. 143·145 yards.

4. The angle of elevation of a tower was 26° 30′, and, 75 yards nearer to it, the elevation was 51° 26′. Required its height and distance. Ans. Height 61·97, dist. 49·294 yards.

5. Measured 149 yards towards a hill, and at the extremities of the line the elevations of its top were 29° 17′ and 39° 25′. Required its height. Ans. 263·02 yards.

PROB. XI. To measure a height which has no level ground before it.

Take two stations C and D, in a vertical plane, and measure CD, and at C take the elevation of D above C, viz. GCD 31° 26′, and the elevations or depressions of the top and bottom of the height, viz. ACF 53° 26′, and BCF 18° 32′, and at D take the elevation of the top ADE 22° 30′, and let CD be 286 feet. Since EDC = DCG, ADC = ADE + DCG = 53° 56′, and DAC = ACF — ADE. The triangle ADC has two angles; and the side CD given, to find AC. Then in the triangle ACB are given $ACB = ACF \pm BCF$, and $B = 90° \pm BCF$, and AC; to find AB.

Ans. Sin. DAC : sin. ADC : : DC : CA, and sin. B : sin. ACB : : CA : AB 271·39.

NOTE 1. If DE be above A, the angle DAC is the sum of ACF and ADE; otherwise it is their difference. Also, in this case ADC is the difference of DCG and ADE; otherwise it is their sum. Also, when F is below B, the angle ACB is the difference of ACF and BCF; otherwise it is their sum.

NOTE 2. If the stations C and D cannot be conveniently taken in a vertical plane, they may be taken anywhere, and then the angles ADC and ACD must be measured with a sextant, and the triangle ACD will give the side AC.

2. At a considerable distance from a hill, I took the elevation of the top of a tower built upon it, 33° 45′; and measuring on level ground 300 feet directly towards the hill, I again took the elevations of the top and the bottom of the tower 51° and 40°. Required the height of the tower.

Ans. 46·666 yards.

3. At a window on a level with the base of a steeple, I took the elevation of its top 40°; and at another window of the same house 18 feet higher, I took again the elevation of the top of the steeple 37° 30′. Required the height of the steeple.

Ans. 210·44 feet.

4. The elevation of the top of a hill at one station was 38° 25′. Another station was taken 450 feet from the first, but neither on a level with it nor in the direction of the hill. At the first station, the line from the other station to the top of the hill subtended an angle of 67° 30′; and at the second, the line from the first to the top of the hill subtended an angle of 74° 48′. Required the height of the hill. Ans. 441·25 feet.

5. I measured directly up a hill 132 yards: there I took the depression of the hill 42°, that of the bottom of a distant object 27°, and that of its top 19°. Required the height of the object. Ans. 28·637 yards.

Prob. XII. To find the distance of a place A, from an inaccessible object B.

First. Let B be visible from A. Choose a station C, from which both A and B can be seen. Measure AC 650 yards, and take the angles BAC 72° 22′, and ACB 78° 37′, with the theodolite. Then ABC is 29° 1′, and sin. B : sin. C :: CA : AB 1313·67 yards.

Secondly. Let B be not visible from A. Choose a station C from which both A and B may be seen, and their distances from it measured. Take the angle ACB 75° 38′, and measure AC 358, and CB 560 feet.

Ans. (BC+CA)918 : (BC—CA)202 :: tan. $\frac{1}{2}$(A+B) 52° 11′ : tan. $\frac{1}{2}$(A—B) 15° 49·7′; whence BAC is 68° 0·7′, and sin. A : sin. C :: CB : BA 585·041.

3. A straight line was measured along the bank of a river 528 feet, and at its extremities the angles contained by it, and straight lines directed to a tree upon the opposite bank, were 62° 40′ and 73° 26′. Required the breadth of the river.

Ans. 648·366 perp. breadth, and 676·444 feet to the nearest station.

4. Straight lines from a station to two places measured 694

and 456 yards, and the angle contained by them was 127° 16′. Required the distance of the one place from the other.

Ans. 1035·773 yards.

5. To find the distance between two trees, I found the angle it subtended at a station to be 55° 40′, and measured from the station to the trees 588 and 672 yards. Required their distance. Ans. 592·967 yards.

PROB. XIII. To find the distance between two places A and B, both of them inaccessible.

Take two stations C and D, such, that from each of them the other station and the places A and B may be seen. Measure CD 1267 links, and at C take the angles BCA 53° 38′, and BCD 34° 50′, and at D take the angles ADC 43° 44′, and ADB 58° 38′.

In the triangle ADC, the angle ACD is 88° 28′, and CAD 47° 48′, and sin. A : sin. C :: CD : DA 1709·69. Also, in the triangle BCD, the angle CDB is 102° 22′, and CBD 42° 48′, and sin. B : sin. C :: CD : DB 1065·14. Lastly, in the triangle ADB are given AD and DB, and the angle ADB. (AD + DB) 2774·83 : (AD — DB) 644·55 :: tan. $\frac{1}{2}$(A + B) 60° 41′ : tan. $\frac{1}{2}$(A — B) 22° 28$\frac{1}{3}$′; whence ABD is 83° 9$\frac{1}{3}$′, and sin. ABD : sin. ADB :: DA : AB 1470·304 links.

2. To find the distance between two steeples A and B, I took two stations C and D, distant 428 yards from one another; and at C took the angles ACB 54° 30′, and BCD 42° 26′; and at D took the angles CDA 40° 44′, and ADB 57° 42′. Required the distance of the steeples.

Ans. 546·704 yards.

3. To find the distance between two places M and P, I took two stations A and B, distant from one another 908·36 feet; and at A took the angles PAM 14° 34′, and MAB 46° 16′; and at B took the angles ABP 96° 44′, and PBM 18° 39′. Required the distance between M and P. Ans. 674·64 feet.

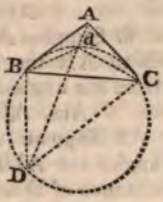
NOTE. If the distance between the objects be known, and the distance between the stations be required, assume 1 or 1000 for the distance between the stations, and with it find the distance between the objects. Then, as the distance found is to the given distance, so is 1000 to the true distance between the stations.

4. Suppose the distance AB 700 feet, and at the station C let ACB be 42° 45′, and BCD 54° 12′, and let the angles at D be ADB 50° 19′, and ADC 57° 33′. Required the distance CD. Ans. 330·04 feet.

5. To find the distance between two lighthouses A and B, I measured the distance between two stations M and R 3370 yards, and at M took the angles AMB 37° 52′, and BMR 91° 27′, and at R the angles ARM 29° 56′, and ARB 40° 27′. Required the distance AB. Ans. 7063·465 yards.

6. At a station C took the angle ACB, subtending a line AB 3291 yards, and found it 4° 35′, and the angle BCD between B and another station D 86° 52′; and at D took the angles ADB 8° 24′, and ADC 70° 23′. Required the distance of the stations from one another. Ans. 3370·425 yards.

Prob. XIV. Given the distances of three places, A, B, C, from one another, viz. AB 317, AC 308, and BC 478 feet, and the angles which these distances subtend at a station D in the same plane with them, viz. ADB 24° 50′, and ADC 27° 44′; to find the distance of the station D from each of the places.



Having drawn the triangle ABC, make at the point C, on the side of BC, opposite to that on which the station D lies, the angle BCd 24° 50′, and at B the angle CBd 27° 44′, and about the triangle BCd describe a circle, and join Ad, meeting the circle again in D, and join BD and DC.

The three sides of the triangle ABC are given to find the angle ABC 39° 25′ 14″; then ABd = ABC ± dBC = 67° 9′ 14″, when A and d are on different sides of BC, or = 11° 41′ 14″, when, as here, A and d are on the same side of BC. Also, the angles of the triangle BCd are given, with the side BC, to find Bd 252·7 feet. Again, in the triangle ABd are given the sides AB and Bd, and the included angle ABd, to find the angles AdB 131° 53′ 53″, and BAd 36° 24′ 53″. Then in the triangle ABD are given the angles and the side AB, to find BD 448·065, and AD 661·738. And in the triangle DBC are given the angles and BC, to find DC 591·57 feet.

2. If A be the place nearest to D, the angle BAd is 46° 47′ 32″: then BD is 550·154, AD 282·25, and CD 528·42 feet.

Note 1. If the given station be within the triangle, as at d, make the angles BCD and CBD equal to the supplements of BdA and AdC.

Note 2. If two of the given places, A and B, be in a straight line with the station D, the distances BC and CA

subtend the same angle BDC. After finding the angle at B, work the triangle DBC.

NOTE 3. If the three places A, B, C, be in a straight line, the first operation will not be required. The rest are the same as before.

3. The three sides of the triangle ABC are AB 280, BC 314, and AC 326 yards; and from the station D without the triangle, the angle ADB was 25° 52′, and ADC 23° 6′, the point C being the nearest to D. Required their distances from D. Ans. AD 586·154, BD 413·41, CD 308·107 yards.

4. Suppose AB 267 feet, BC 209, and AC 346, and at the point D, within the triangle, the angle ADC is 128° 40′, and ADB 91° 20′. Required the distances of D from the angles.

Ans. AD 104·05, BD 189·33, and DC 178·85 feet.

NOTE. When D is in one of the sides, describe a segment on BC containing the given angle.

5. Suppose AB 122·4, BC 74, and AC 82 chains, and at D in AB, produced beyond B, the angle ADC is 22° 45′. Required the distance of D from the angles.

Ans. AD 181·8, BD 59·4, and CD 125·4 chains.

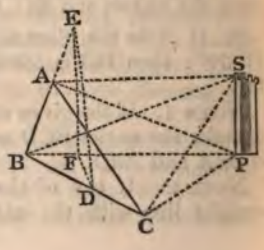
6. Suppose AB 1234, BC 873, and AC 632 yards, and at D in AB the angle ADC is 120°. Required its distance from the angles.

Ans. AD 226·12, BD 1007·88, and CD 487·84 yards.

7. Suppose AB 138, BC 224, and AC 326, and at D the angles are ADB 7° 22′, and ADC 19° 58′. Required the distance of D from the angles.

Ans. AD 510·96, BD 385·286, and DC 204·87.

PROB. XV. Given the angles of elevation of a tower PS, taken at three stations A, B, and C, on a level plane, no two of which are in the same vertical plane with the tower, viz. PAS 20° 10′, PBS 18° 50′, and PCS 34° 30′, and also the distances between the stations AB 324, BC 568, and AC 672 yards; to find the height of the tower.



Make the triangle ABC, of which AB is 324, BC 568, and AC 672, and make BE = BC, and BD = BA, and join ED, and upon it make the triangle EDF on either side of DE, so that BE : EF : : cot. PBS : cot. PAS, and BD : DF : : cot. PBS : cot. PCS; or make EF 527·494, and DF 160·79, and join BF, and

make the angle BAP = BFE. Then erect PS perpendicular to the plane ABP, and in the plane passing through AP and PS make the angle PAS 20° 10′, and PS will be the tower required.

Join PC, CS, BS, the triangles APB, FBE, being similar, AP : PB :: FE : EB :: cot. SAP : cot. SBP, therefore SBP is 18° 50′; also PB : BE = BC :: BA = BD : BF, therefore the triangles PBC and FBD are similar; and BP : PC :: BD : DF :: cot. PBS : cot. PCS, therefore PCS is 34° 30′.

In each of the triangles EBD, EFD, are given the three sides, to find the angles BED 28° 45′ 30″, and FED 6° 47′ 26″; and their difference 21° 58′ 4″, or their sum 35° 32′ 56″, is the angle BEF, from which, with the sides BE and EF, the angle BFE or BAP is found in the first case to be 89° 48′ 7″, and in the other 78° 48′ 22″. Therefore AP is 866·108 or 546·676, and PS 318·094 or 200·78.

2. Let AB be 326, BC 584, and AC 683, and the angles of elevation SAP 30°, SBP 26°, and SCP 23°; to find PS.

Ans. PS is 952·14 or 168·642.

3. Let AB be 80, BC 119, and AC 140 yards, and the elevation at A 50°, at B 60°, and at C 55°. Required the height of the object D. Ans. 96·4 feet.

4. Let AB be 60, BC 72, and AC 132 feet, and the elevations of S at A 30° 48′, at B 40° 33′, and at C 50° 23′. Required the height of S. Ans. 94·84 feet.

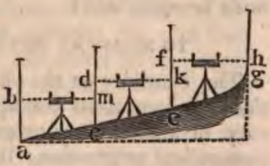
5. Let AB and BC be each 84 feet, and the points A, B, C, in a straight line, and the elevation at A 36° 50′, at B 21° 24′, and at C 14°. Required the height of the object.

Ans. 53·96 feet.

OF LEVELLING.

When the altitudes of the several parts of an irregular ascent are to be determined, a spirit-level with telescopic sights is to be used.

PROB. XVI. To find the height of g above a.



Erect a pole *ab* at *a*, and another *cd* at a convenient distance. Place the level between them, and, directing the sights to the pole *ab*, cause the point *b* to be marked on it; then direct the sights to the pole *cd*, and on it mark *m*. Next erect a pole nearer to *g*, as at *e*, and place the level between it and the pole *cd*, and mark upon them, as before, the points *d* and *k*; and proceed in this way to *g*.

To find the height of g above a, take the sum of the heights ab, cd, &c. got by looking towards a, and from it subtract the sum of the heights cm, ek, &c. got by looking towards g: the remainder is the height of g above a. In like manner the heights of c, e, &c. above a are got. If the horizontal distance between a and g be required, add bm, dk, &c.

To find the height of any point c in a regular ascent: The distance ag is to ac, as the height of g above a to the height which c ought to have above a.

It is not necessary to place the poles in the same direction with ab and gh, but it is necessary to erect them perpendicular, or nearly so.

NOTE. When the distance between the poles ab and cd is very great, the line bm will differ a little from the true level; for bm is a tangent to a great circle of the earth, passing through the centre of the instrument, and the true level is the arc of that circle between the poles ab and cd. The correction may in general be neglected: for a mile it is 7·96 or 8 inches; and for other distances from the instrument, the correction varies as the square of the distance.

1. Let the heights on the poles taken by looking down the eminence be 11, 8, 5, 6, 4, and those taken by looking up be 5, 3, 1, 4, 6 feet. Required the height of the eminence.

Ans. 15 feet high.

2. Let the heights taken by looking down be 10, 11, 7, 5, 8, 4, 9, and those taken by looking up be 3, 5, 2, 6, 4, $5\frac{1}{2}$, $3\frac{1}{2}$ feet. Required the height of the eminence. And, supposing the sloping distance from the bottom to the top to be 346 feet. Required the height in a regular slope at the distance of 136 feet from the bottom.

Ans. 25 feet high in all, and, at 136 feet, 9·8266 feet.

3. A hollow in a road, of which the depth on the lowest side is 56 feet, and on the upper 74, and the width at the top of the lower side is 234 feet, and at the bottom 87, and half-way up 172 feet, is to be filled up from the road on the upper bank, so as to form a regular slope. How much of the road must be excavated? Ans. 1263·13 feet.

TO MEASURE HEIGHTS BY THE BAROMETER.

The elasticity or the density of the air is as the weight of the superincumbent atmosphere; and therefore, if the heights vary in arithmetical progression, the densities will vary in geometrical progression; that is, the height is as the logarithm of the density. It has been found by experiment, that the module of the barometrical logarithms is 10,000 times that of the common logarithms; wherefore, if B be the height of the

mercury at the lower station, and b that at the higher, and h the difference of the heights of the stations, then $h = 10,000 \times$ (com. log. B — com. log. b) expressed in fathoms. But this formula is true only upon the supposition that the temperature of the air is $32°$, and that it is the same at both stations; neither of which is exactly true.

It is found by experiment, that quicksilver expands about $\frac{1}{10000}$ part of its bulk for every degree of Fahrenheit's thermometer. Let r be the temperature at the lower station, and r' that at the higher, as indicated by the thermometer attached to the barometer, then $b + \frac{r-r'}{10000}b$ will be the height of the mercury at the higher station, when reduced to the same temperature with that at the lower station; and thus $h = 10000 \times \left(\log. \mathrm{B} - \log. \left(b + \frac{r-r'}{10000}b\right)\right)$.

Again, the air expands nearly ·00245 of its bulk for every degree of Fahrenheit's thermometer. Let t be the temperature of the air at the lower station, and t' that at the higher, as indicated by a thermometer in the open air, then $\frac{1}{2}(t+t')$ may be taken for the mean temperature; and therefore the former formula has to be multiplied by $\cdot 00245 \times \left(\frac{t+t'}{2} - 32\right)$ for an additional correction.

Prob. XVII. To find the height of one place above another.

From what has been shown, the complete formula will be $h = 10000 \times \left(\log.\ \mathrm{B} - \log. \left(b + \frac{r-r'}{10000}b\right)\right) \times \left(1 + \cdot 00245 \times \left(\frac{t+t'}{2} - 32\right)\right)$, which, expressed in words, gives the following

Rule. Divide the difference of the heights of the attached thermometer by 10000, and add 1 to the quotient, and add the logarithm of the sum to the logarithm of the height of the barometer at the highest station, and subtract the sum from the logarithm of the height of the barometer at the lower station: the remainder, multiplied by 10000, will give the approximate height. Take the difference between $32°$ and half the sum of the heights of the detached thermometer, and multiply it by ·00245; and if the half sum of the heights be greater than $32°$, add the product to 1, otherwise subtract; and the sum or remainder, multiplied by the approximate height, will give the true height.

NOTE. This method of finding heights is convenient, but it is not very accurate.

1. Suppose the height of the mercury in the barometer at the bottom of the hill to be 29·56 inches, and at the top 28·27 inches, and the temperature of the mercury 63° and 54°, and the temperature of the air 56° and 48°. Required the height of the hill.

Ans. $\frac{63-54}{10000} = \cdot 0009$ and $10000 \times (\log.\ 29\cdot 56 - \log.\ 28\cdot 27 - \log.\ 1\cdot 0009) = 10000 \times (1\cdot 4707044 - 1\cdot 4513258 - 0\cdot 0003907) = 10000 \times \cdot 0189879 = 189\cdot 879$ fathoms $=$ 1139·274 feet, the approximate height. Also, $\frac{1}{2}(56+48) - 32 = 20$, and $1 + 20 \times \cdot 00245 = 1\cdot 0489$; therefore $1139\cdot 274 \times 1\cdot 0489 = 1195\cdot 098$ feet, the true height.

2. Let the height of the barometer at the lower station be 29·57, and at the higher 28·7 inches, the height of the attached thermometer at the lower 55·28°, and at the higher 51·75°, and the temperature of the air at the lower 54°, and at the higher 50·5°. Required the elevation. Ans. 807·117 feet.

3. Let the heights of the barometer be 29·4 and 25·19 inches, the attached thermometer 50° and 46°, and the temperature of the air 45° and 39°. Required the elevation.
Ans. 686·458 fathoms.

4. Let the heights of the barometer be 29·89 and 26·27 inches, the attached thermometer 56·5° and 42·75°, and the temperature of the air 55·25° and 43°. Required the elevation. Ans. 3467·783 feet.

PROB. XVIII. To measure distances by sound.

RULE. Multiply the time the sound takes in seconds by 1142: the product will be the distance in feet.

NOTE. Sound in common air moves uniformly at the rate of 1142 feet in a second. Cold, and uneven surfaces, retard its motion a little, and heat accelerates it in a small degree.

1. I observed the flash of a gun 30 seconds before I heard the report. How far was it distant from me?
Ans. $30 \times 1142 = 34260$ feet.

2. I observed a flash of lightning, and after 6 strokes of my pulse I heard the thunder, and my pulse makes 68 strokes in a minute. How far was the thunder distant from me?
Ans. 1 mile 255 yards.

3. How long, after firing a gun, will it be till the report is heard at the distance of 8 miles? Ans. 37 seconds.

4. A person standing on the bank of a river heard the echo of his voice reflected from a rock on the opposite bank, in 4

seconds after he uttered it. What is the breadth of the river? Ans. 2284 feet.

PROB. XIX. To measure a height by the descent of a stone, &c.

RULE. Multiply the square of the time of descent in seconds by $16\frac{1}{12}$: the product will be the height in feet.

To find the time of descending. Divide the height in feet by $16\frac{1}{12}$, and the square root of the quotient will be the time in seconds.

NOTE. A heavy body descends $16\frac{1}{12}$ feet in the first second of time, and the spaces descended are as the squares of the times.

1. A stone takes 3 seconds in falling from the top of a tower to the ground. What is the height of the tower?

Ans. $3 \times 3 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

2. In what time will a stone dropt from the height of 579 feet reach the ground? Ans. 6 seconds.

3. What is the height of a precipice, when a stone takes 7 seconds in falling from the top to the bottom?

Ans. $788\frac{1}{12}$ feet.

4. I reckoned 7 strokes of my pulse during the falling of a stone from the top of a rock. What height did it fall, the pulse beating 70 times in a minute? Ans. 579 feet.

5. While a stone descended from the top of a tower, a pendulum 10 inches long made 8 vibrations. Required the height.

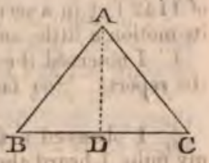
Ans. 263 feet.

TO SURVEY FIELDS.

PROB. XX. To survey a triangular field ABC.

First, with the Chain only. Measure the three sides by Prob. I.

Secondly, with the Chain and Cross. Measure along BC by Prob. I., and with the cross find the point D, where the perpendicular from A meets BC, by Prob. VI. Write down the measures of BD, BC, and DA.



Thirdly, with the Theodolite and Chain. Measure one angle ABC by Prob. III., and the containing sides AB and BC by Prob. I. Or measure BC by Prob. I., and two angles ABC and ACB by Prob. III. From these measures the plan may be easily drawn by Prob. XIX. XX. or XXI. of PRACTICAL GEOMETRY; and the area may be found by Prob. IV. V. or VI. of MENSURATION.

1. In a triangular field I measured the base 856 links, and found the extremity to be the foot of the perpendicular upon it, which I measured 672 links. Required the content.

Ans. 2 acres 3 roods 20 perches 5 yards 5·53 square feet.

2. In measuring the base of a triangular field, I found the foot of the perpendicular 256 links from its extremity, the base 927 links, and the perpendicular 582 links. Required the area. Ans. 2 acres 2 roods 31 perches 18 yards 4·4 feet.

3. I measured an angle of a triangular field 73° 24′, and the sides containing it 688 and 492 links. Required the plan of the field, and the area.

Ans. 1 acre 2 roods 19 perches 15 yards 4 feet.

4. I measured one side of a triangular field 1268 links, and took the angles at its extremities 57° 36′ and 62° 24′. Required the area.

Rule. Add the sines of the given angles and the log. of the side, and subtract the sine of the third angle, or of the sum of the given ones, to get the perpendicular = 1095·55.

Ans. 6 acres 3 roods 31 perches 9·8591 yards.

5. The three sides of a triangular field are 1275, 987, and 642 links. Required the area.

Ans. 3 acres 17 perches 24 yards 3·1068 feet.

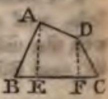
Prob. XXI. To survey a field contained by four sides.

Fig. 1.

First, with the Chain only. Measure the four sides and a diagonal BD by Prob. I.

Secondly, with the Chain and Cross. Measure along a diagonal BD by Prob. I., and, with the cross, find by Prob. VI. the points E and F, upon which the perpendiculars fall from A and C, and write the lengths of BE, BF, BD, AE, and CF.

Fig. 2.



Or measure the longest side BC, marking E and F the places of the perpendiculars, and measure AE and DF.

Thirdly, with the Theodolite and the Chain. Place the theodolite at B (fig. 1,) and take the angles ABD and DBC by Prob. III., and measure the diagonal BD by Prob. I., and again at D take the angles ADB and BDC. Or take the angle ABC, and measure the four sides.

If the angle ABC cannot be measured conveniently within the field, fix a pole G in the direction of either side AB, extended beyond B, and measure the angle CBG, which, subtracted from 180°, will give ABC.

Fourthly, with the Plane-Table and the Chain. Place the table at one of the angles B, from which all the other angles may be seen, and turn it round till the needle points to the fleur-de-lis, and there fix it. Fix also a pin in some part of the paper to represent B. Apply the fiducial side of the index to the pin, and turn it till the angle A is seen through the sights. Draw a line from the pin in that direction. Measure BA, and by the scale on the index lay it on that line from B to A. Next turn the index till the angle D is seen through the sights, and draw a line in that direction, and on it lay the length of BD. Lastly, draw a line in the direction of C, and on it lay BC, and join CD and DA. In the same manner any field may be surveyed by the plane-table, when an angle can be taken, from which all the other angles of the field are seen.

1. I measured along the diagonal BD, (fig. 1,) and at E, 118 links from B, was the foot of the perpendicular AE 318, and at F, 527 links from B, was the foot of the perpendicular CF on the opposite side of BD, 426 links: the whole length of the diagonal BD was 968 links. Required the plan and the area.

Ans. Area 3 acres 2 roods 16 perches 4 yards 5·8176 feet.

2. I measured along BC the longest side of a four-sided field ABCD, (fig. 2,) and at E, 125 links from B, was the foot of the perpendicular AE, which measured 624 links, and at F, 635 from B, was the foot of another perpendicular FD 462 links: the whole length of the side BC was 1274 links. Required the plan and the area.

Ans. Area 4 acres 2 roods 21 perches 20·0376 yards.

3. I measured an angle ABC of a quadrilateral field 128°, and the four sides AB 536 links, BC 843, CD 634, and AD 936 links. Required the plan and the area.

Ans. Area 4 acres 2 roods 26 perches 16 yards $5\frac{1}{4}$ feet.

4. I measured the diagonal BD of a four-sided field 1462 links, and at its extremities I took the angles which it made with the sides, viz. ABD 48° 20′, CBD 41° 26′, ADB 29° 40′, and BDC 38° 44′. Required the plan and the area.

Ans. 8 acres 2 roods 4 perches 28 yards $3\frac{1}{4}$ feet.

5. In taking the plan of a quadrilateral field by the plane-table, I found the straight side AB to lie N. 73° E., and to measure 568 links; the diagonal AC to lie S. 83° E., 978 links; and the side AD to lie S. 47° E., 734 links. Required the plan and the area.

Ans. 3 acres 38 perches 9 yards 3·071 feet.

PROB. XXII. To survey any field with the chain.

First, with the Chain only. Measure all the sides of the field, and then the diagonals BF, FC, FD. From these the field may be drawn upon paper by Prob. XXVIII. of PRACTICAL GEOMETRY, and its area may be found by Prob. XI. of MENSURATION OF SUPERFICIES.

1. In a six-sided field I measured all the sides, viz. AB 583 links, BC 324, CD 456, DE 892, EF 728, and AF 477 links, and from F measured the diagonals FB 897, FC 723, and FD 948 links. Required the plan and the area.

Ans. 7 acres 12·9 yards.

Secondly, with the Chain and Cross. Divide the field by diagonals into as many trapezes as possible, and the remainder will consist of one or more triangles. Thus the field ABCDEF may be divided into two trapezes ABCF and CDEF, by joining CF. These may be surveyed as in the last Problem.

2. In a heptagonal field I measured along the northernmost diagonal BG, and at 207 links from B found the foot of a perpendicular above it AH, which measured 272; and at 578 from B found the foot of a perpendicular under it FK, which measured 498; the diagonal BG 928. From F, I measured along a diagonal FC, and at 488 from F was at the foot of the perpendicular from B, which measured 587, and the diagonal FC 896. Then, from C, I measured along a diagonal CE, and at 498 from C was the foot of an under perpendicular ND 630, and at 688 from C was at the foot of a perpendicular FM 574 links; the diagonal CE was 1093 links. Required the plan and the area.

Ans. 12 acres 3 roods 5 perches 5 yards 5·965 feet.

NOTE. If a perpendicular, as E*p*, upon a diagonal DF, fall without the field, and it be inconvenient to measure it in that situation, the other diagonal CE, with the perpendiculars upon it, may be taken; or the two triangles DEF, CDF, may be measured separately.

3. In a hexagonal field ABCDEF, I measured along the diagonal BF, and, at 328 links from B, I was at the foot of the perpendicular AG, which measured 286, and the diagonal BF was 536; but had to measure 127 links farther without the field, to come to the foot of the perpendicular EH on the opposite side of BF, which measured 453. Again, measuring along the diagonal EC, I found, at 386 from E, the foot of the perpendicular DK, which measured 496; and, 674 from E, found the foot of the perpendicular BL, which mea-

sured 436; the whole length of the diagonal EC was 895 links. Required the plan and the area.

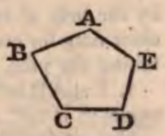
Ans. 6 acres 24 perches 5 yards 8·1432 feet.

Thirdly. In fields not very large, it will be sufficient to measure one diagonal, and the perpendiculars upon it from all the other angles.

4. Suppose the distances of the perpendiculars from A to be 50, 145, 220, 295, 380, 475, and 655, the whole line AD being 725 links, the second and sixth distances reach to perpendiculars on the right hand, and the rest to those on the left hand. Also the perpendiculars on the right are 75 and 150, and the others in their order are 110, 135, 85, 275, and 185 links. Required the plan and the area.

Erect perpendiculars upon AD, at their proper distances from A; and, having made them of their proper length, the plan is drawn by joining their extremities. The area is easily found by Prob. IV. and VII. of Mensuration of Superficies. Ans. 1 acre 3 roods 5 perches 1 yard 7·335 feet.

Prob. XXIII. To take the plan of a field by going round it.



First, with the Plane-Table. Place the table at a corner A, and fix it when the needle points to the fleur-de-lis, and take a point A on the paper. Direct the index from the assumed point to the corner E of the field, and draw a line; then direct the index to B, and draw another line. Measure the lines in the field from A to B and from A to E, and lay these lines on the paper. Place the table at B, and, laying the index along BA on the paper, turn the table about till A is seen through the sights: the needle ought then to point to the fleur-de-lis. Direct the index to the corner C of the field, and draw a line, on which lay the length of BC. In the same manner are to be laid down the position and the lengths of the other sides CD and DE, and the last line will terminate at E on the paper, if no error has been committed.

Secondly, with the Theodolite. Place the instrument at the corner A of the field, and, having turned it till the needle points to the fleur-de-lis, take the bearing of one of the sides, as AE; then observe the angle EAB, and measure AB. Again, place the theodolite at the corner B, and observe the

angle ABC, and measure BC. And proceed in this way to take all the angles and to measure the sides.

Add all the angles together, if they be interior; but if any of them be exterior, add the difference between it and 360°: the sum should be equal to 180°, multiplied by the number of sides, wanting two.

If the interior angles cannot be taken, let the exterior be taken by extending the direction of the sides. The sum of all the exterior angles should be 360°; but if any of the corners point inward, add 180° to 360° for every such angle, and the sum should be the sum of the angles.

The things measured for laying down the plan of a field will always be sufficient for finding its content, but they will not always afford the shortest method. Thus, in taking the plan of the pentagonal field ABCDE by measuring the sides and angles, if we draw diagonals AC and CE, we can find the area of the triangle ABC from the sides AB and BC and the angle B, and the triangle CDE from the sides CD and DE and the angle D; but then we have nothing given in the triangle ACE from which to find its area. We must therefore find, by trigonometry, in the triangle ABC, the angle ACB and the base AC, and in the triangle CDE, the angle DCE and the base CE; and these two angles, subtracted from BCD, will give the angle ACE, from which, with the sides AC and CE, we can find the area of the triangle ACE. And thus, by the help of trigonometry, we may find in every case sufficient data for computing the area from the things measured for taking the plan. Shorter methods are given afterwards.

1. Let AB be 750, BC 810, CD 628, DE 598 links, and the angles at B 72°, at C 136°, and at D 122°. Required the area.

The angles will be found to be ACB 50° 58′ 11″, DCE 28° 13′ 23″, and ACE 56° 48′ 26″, and AC 918·23, and CE 1072·32 links.

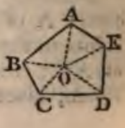
Ans. Area 8 acres 2 roods 16 perches 6 yards 4·283 feet.

2. In a six-sided field ABCDEF, let AB be 482, BC 586, CD 760, DE 812, and EF 910 links, and the angles at B 96°, at C 132°, at D 146°, and at E 106°. Required the area.

Ans. Area 15 acres 3 yards 5·122 feet.

PROB. XXIV. To survey a field from a station within it.

The station must be chosen such, that all the angles may be seen from it.



First, with the Plane-Table. Place the table at O, from which all the corners may be seen, and turn it to bring the needle to the fleur-de-

lis; and on the paper take a point O, to represent the station. Direct the index from O to the corner A, and draw a straight line to represent OA in the field. Draw, in the same manner, lines to represent OB, OC, &c. Then measure from the station to A, B, C, &c. in the field, and lay them on their representatives, and join their extremities.

Secondly, with the Theodolite. Place the instrument at the station O, and, putting the needle to the fleur-de-lis, take the bearing of OA. Next observe the angles AOB, BOC, &c., which, added, should amount to 360°. Then measure straight lines from O to A, B, C, &c.

1. Suppose OA 798, OB 459, OC 434, OD 852, and OE 912 links, and the angles at O, AOB 74°, BOC 38°, COD 102°, DOE 82°, and EOA 64°. Required the area.

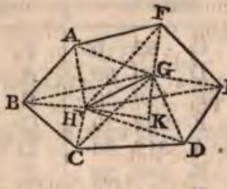
Ans. 11 acres 1 rood 8 perches.

2. In a heptagonal field I found the angles at the instrument to be 67°, 43°, 84°, 56°, 27°, 51°, and 32°, and the distances of the angles from the instrument to be 528, 632, 916, 478, 732, 830, and 816 links. Required the plan and area. Ans. Area 12 acres 1 rood 6 perches 12·07 yards.

Prob. XXV. To survey a field from two stations.

The stations must be such, that all the objects to be laid down on the plan may be seen from them both, and that the angles which they make with the line joining the stations may not be too small.

First, with the Plane-Table. Place the table at one of the stations, and the needle to the fleur-de-lis, and take a point G on the paper to represent that station, and direct the sights of the index from it to the other station, and draw GH, and on it lay the distance between the stations from G to H.



Direct the sights from G to the corner A, and draw GA with a black-lead pencil, and upon any part of it place the letter A. Again direct the sights from G to the corner B, and draw GB, and on it write B. In the same manner draw GC, GD, &c.

Remove the table to the second station, and turn it till the needle points to the fleur-de-lis; then the index, laid on HG of the paper, will point to the former station. Direct now the sights from H to the corner A, and draw HA, which will meet the line GA in the point representing that corner, at which place A, and erase the former A. In the same manner draw HB, meeting GB in B, and so on; then join AB, BC,

&c. In the same way the position of any other thing, as the house K, may be determined by drawing GK towards it when the table is at G, and HK towards it when the table is at H.

Secondly, with the Theodolite. Place the instrument at the first station G, and turn it till the needle points to the fleur-de-lis, and take the bearing of the station H, and measure GH. Then take the angle HGC, then CGD, DGE, &c., and lastly BGH. Remove the instrument to the second station H, and bring the needle to the fleur-de-lis; then the station G ought to bear upon the point opposite to that upon which H bore from G. If it does, then take first the angle GHF, then FHA, AHB, &c., and lastly EHG. The sum of the angles taken at each station ought to be exactly 360°.

Every thing else which is to be put in the plan must be surveyed in the same way, by taking at G the angle between GH and the line from G to it, and the same at H. All these observations must be placed in a field-book.

When the whole cannot be seen at two stations, more stations must be taken. The lines between the stations must be measured, and the angles taken as before. But care must be taken to determine the position of each of the lines joining the stations.

1. Required the plan and the area of a field from the following

FIELD-BOOK.

Angles at G.	Angles at H.	Remarks.
C 22° 0′	F 20° 0′	GH bears S. 67° 30′ W.
D 86 30	A 72 0	1038 links.
E 146 30	B 145 0	Corner of a house at K.
F 232 30	C 243 0	Angles { at G 50°.
A 313 30	D 317 0	{ at H 323°.
B 348 30	E 344 0	
H 360 0	G 360 0	

In this field-book, the angles at G are marked as taken with the theodolite when placed at that station. The sights, when at the beginning of the degrees, were directed to the station H, and the instrument fixed there. Then the moveable index was turned to C, and cut off 22° for the angle HGC, which, in the field-book, is marked C, the other two letters being found at the top; then it was turned to D, and cut off 86° 30′ for the angle HGD; and the difference of these two is the angle CGD. It was then turned to E, and cut off 146° 30′ for the angle HGE; and so on all the way

round. In the same way the angles were taken at H, both for determining the corners of the field and for finding the corner of the house at K.

In calculating the areas of fields surveyed from more than one station, it is necessary to calculate, by trigonometry, the length of all the lines drawn from one of the stations to the angles; and for this purpose we have, in every triangle of which GH is a side, all the angles and this side to find the other side; after which the area is found as in the preceding problem. Here the distances from G are GA 1123·3, GB 1493·1, GC 1409·73, GD 917·43, GE 951·47, and GF 660·743 links; from which the areas of the triangles AGB, BGC, CGD, DGE, EGF, and FGA, are to be calculated.

Ans. 27 acres 5 perches 25 yards 3·47 feet.

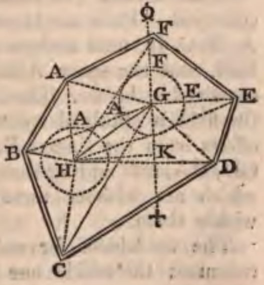
2. Required the plan and the area of a field from the following

FIELD-BOOK.

Angles at P.		Angles at R.		Remarks.
F	3°	A	6°	PR bears S. 22° 30′ E. 1827 links.
E	28	H	24	
D	49	G	64	
C	65	F	186	
B	132	E	228	
A	197	D	271	
H	247	C	319	
G	320	B	342	
R	360	P	360	

Ans. Area 100 acres 1 rood 19 perches 21 yards 1·4 foot.

PROB. XXVI. To draw the plan of the field upon paper from the field-book.



Draw a faint line up and down the paper to represent the meridian, the upper end the north, and the under end the south. Using the data given in Ex. 1, Prob. XXV., in this line take a convenient point G for the first station. On the south side of G make an angle of 67° 30′ towards the left hand, which will give the position of GH; and take 1038 from any convenient scale, and lay that extent from G to H,

to get the station H. The best protractor for laying down the angles is a circular one, divided into 360°. Place the centre at G, and the beginning of the degrees on GH. Make a mark at 22°, and at it write a faint C; make another mark at 86° 30′, and there write a faint D, and so on all the way round; and draw faint lines from G to the marks. Next place the centre of the protractor at H, and the beginning of the degrees on GH; and at 20° make a mark, and write F; at 72° make a mark, and write A, and so on; and draw lines from H through the marks. The lines from G and H, through the points where the same letter is written, must be drawn out till they meet, and their intersection is at the angle to which that letter belongs. Thus GA and HA will meet in the angle A, GB and HB will meet in the angle B, &c. After this join AB, BC, &c. for the boundaries of the field.

If the protractor be a semicircle, then, after laying down the angles less than 180°, the protractor must be laid on the other side of GH, and 180° taken from each of the remaining angles before they are laid down.

PROB. XXVII. To survey fields with crooked boundaries.

The boundaries of fields are seldom straight lines, and therefore surveyors generally erect poles near the corners of the ground to be surveyed, and conceive these poles joined by straight lines. This constitutes the body of the field; and the parts between these lines and the boundaries are considered as offsets, and their areas found separately.

The points, therefore, which, in the preceding problems, were called angles or corners, are to be considered only as the places of these poles, and the fields surveyed as contained by the lines joining them; and to complete the survey, the situation and distance of the boundaries from these lines must be found.

1. Let EIMP be a field to be surveyed. Poles are erected at A, B, C, D, near corners of the field, and the space ABCD is surveyed as before. The rest of the field is obtained by taking offsets from the lines AB, BC, CD, DA, and adding the spaces which are without these lines, and taking away the spaces within them.

The field-book for such a survey must consist of three columns: the middle one contains the distance measured along

ines AB, BC, the other two offsets, accord- are on the right the main line. rpose it is best the bottom of ok, and to write hat the offsets ht side of the may be placed -hand column, ets on the left left-hand co- s, in measuring B, the offset measures 106 the left hand the beginning therefore write niddle column, m, and opposite e left-hand co- e 106. Then along AB, the be found, upon perpendicular F: this is 284 A, and fF is therefore write middle column, posite to it in d column. A- 2 links from A,

FIELD-BOOK.

Left off-sets.	Main lines.	Right off-sets.
	AC, S. 60° 25′ E. 1896.	
	844	Including offset to cor.
86	746	Close to A.
152	688	
	594	
	462	200
D	64	90
	1410	D Γ
	1362	92
	924	196
	744	
146	600	
C 48	0	
108		C Γ
104	912	
264	508	
84	152	
B 70	0	
128		B Γ
94	1672	
172	1166	
	752	
	530	108
	442	
200	284	
A 106	0	
To left.		To right.

B crosses the boundary-line FG; therefore write middle column, and in the adjacent columns draw s in the direction of the straight line FG nearly, t position of it is not required at this stage of the t 530 the perpendicular from G meets AB, and place therefore 530 in the middle column, and e to it in the right-hand column.

in this way to B, where, besides the offset, BI is nd placed in the left-hand column, with the mark that it is not perpendicular. At the same place -hand column is placed the mark Γ, to show that veyor turns to the right hand. This finishes the *the line AB*, and a line is drawn across the book

to separate it from the next line. Proceed in the same way from B to C, from C to D, and from D to A.

The position of any one of the lines, as AC, being found with the compass, it will determine the position of the whole. But in using the compass, the variation should be allowed; and great care ought to be taken lest the needle be attracted by some metallic substance in its neighbourhood.

Ans. Area 14 acres 2 roods 19 perches $22\frac{1}{3}$ yards.

(2.) FIELD-BOOK.

Left off-sets.	Main lines.	Right off-sets.
Diagonal AC, N. 28° W. 760 links.		
0	660	
30	450	
D 0	400	
0	490	D Γ
10	400	
40	300	
55	200	
C 20	50	
	635	0 C Γ
	500	25
	400	30
	300	
50	200	
B 40	100	
0	395	B Γ
20	350	
35	300	
45	250	
50	200	
30	100	
A 15	50	

Ans. 3 ac. 28 per. 7·038 yds.

(3.) FIELD-BOOK.

Left off-sets.	Main lines.	Right off-sets.
Diagonal AC, S. 56° E. 1560 links.		
	1350	
0	1200	
40	900	
20	750	
60	550	
85	400	
70	350	
D 35	200	
0	800	D Γ
34	700	
	500	
	350	80
C	200	60
B	1100	C Γ
0	912	B Γ
40	800	
	750	
	680	50
	600	
90	450	
A 50	340	

Ans. 10 ac. 3 ro. 10 per. 17 yds. 5·558 feet.

Lay down the plans of the following properties from the field-book for the three examples, and calculate their contents.

Fig. 1.

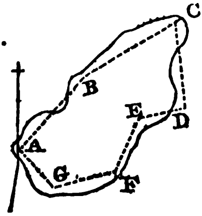
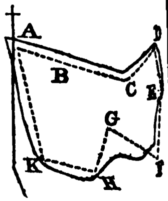


Fig. 2.



(4.) (Fig. 1.)

Diagonals.
BD 1100
BE 720
BF 1080
AF 1000
B bears N. $37\frac{1}{4}$° E.

[illegible]	510	A ┌
[illegible]	360	
[illegible]	0	
[illegible]	612	G ┌
[illegible]	320	
[illegible]	0	
[illegible]	600	F ┌
[illegible]	256	
[illegible]	0	
[illegible]	480	E ┐
[illegible]	220	
	114	
	0	30
┌	920	36
	826	78
	560	340
	356	90
	281	
[illegible]	180	
[illegible]	0	
[illegible]	900	C ┌
[illegible]	728	
[illegible]	560	
[illegible]	256	
[illegible]	0	
[illegible]	1040	B ┌
[illegible]	980	
	826	
	673	56
	522	
[illegible]	443	
[illegible]	156	
[illegible]	0	A

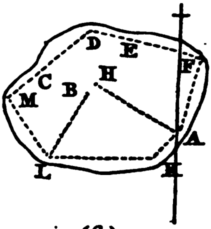
15 ac. 5 per. 15 yds.
3·394 feet.

(5). (Fig. 2.)

Diagonals.
CE 620
CF 1000
CG 610
GB 850
GK 710
BK 940
AK bears S. 11° E.

20	1150	A ┌
25	680	
35	420	
50	0	
60	580	K ┌
90	500	
150	300	
100	0	
89	470	H ┌
130	260	
200	0	
400	800	G ┐
380	630	
220	480	
36	230	
	153	
	110	25
	0	40
F ┌	760	50
	640	78
	520	115
	380	85
	200	40
	86	
30	0	
30	420	E ┌
35	320	
30	100	
20	0	
25	500	D ┌
89	360	
72	150	
30	0	
40	730	C ┐
150	540	
110	210	
30	0	
20	450	B ┐
70	250	
30	0	A

Ans. 18 ac. 1 ro. 23 per.
25 yds.



(6.)

 15 20	2180 626 426 0	A 15 H 10	S. 59° E.
20 20 To houses.	1610 1590 0	10 B ⌈ L	N. 29° E.
A ⌈ 180 9	2050 1969 1000 0	15	S. 13° W.
51 120 20	1380 600 0	F ⌈	S. 77° E.
20 24 10	750 500 0	E ⌈	S. 85° E.
10 500 400 300	1400 1000 700 400 25 0	D ⌈ 20	N. 51° E
15 10	655 0	C ⌈	N. 45° E.
10 350 20	1450 600 0	M ⌈	N. 31° W.
20 220 10	2280 1400 0	L ⌈	N. 85° W.
10 100 20	640 400 0	K ⌈ A	N. 36° W.

Ans. 89·26 acres.

PROB. XXVIII. To take an extensive survey.

Choose for stations the most eminent places, from which the principal parts of the survey may be seen. Particularly choose such eminences as lie near the boundaries. Take the angles which these stations make with one another with great accuracy, and measure carefully in a straight line the distances from station to station, marking the places where the lines pass ditches, roads, rivulets, &c., and take offsets to near objects, leaving in the ground a mark at every place where you marked the distance in the field-book, distinguishing these marks by letters or figures, that they may not be mistaken for one another. In this way you will obtain the situation of the principal parts. Then take other stations within these, and measure the distances as before. And thus divide and subdivide the survey, till you come to single fields, which may be measured by some of the preceding methods.

The longer the distance is between the stations, if accurately measured, the more correct will the work be; but this cannot be ascertained by a single measurement, without using various methods of determining it. At the same time, an error in these primary distances affects the whole survey; and therefore every care ought to be taken to prevent it.

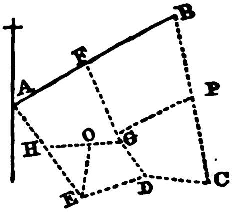
After the principal parts of the survey are laid down accurately, so as to have the whole divided into small compartments, these may be filled up by the plane-table, one by one.

In laying down the plan, proceed in the same way, first laying down the principal distances and the boundaries, and then the interior parts as they are surveyed; and in filling up the particular departments, care must be taken to lay down the boundaries of parishes, estates, farms, &c. and to point out the particular situations of towns, villages, churches, gentlemen's seats, towers, farm-steads, also rivers, lakes, ponds, woods, plantations, rocks, precipices, and all the eminences, mines, pits, quarries, and in general every thing which can contribute to give a proper understanding of the nature of the survey. All these must be neatly sketched and properly coloured, and the names of the places are to be printed in them.

1. I took two stations near a road, of which B lay from A, N. 61° E. 1850 links; and from A took the bearings of the eminences C, S. 70° E., D, S. 62° E., and E, S. 36° E., and at B took their bearings C, S. 14° E., D, S. $6\frac{1}{2}$° W., and E, S. 26° W. Required their distances from the stations, and their bearings and distances from one another.

Ans. BC 1684·14, AE 1201·788, CD 596·64, and D 753·41 links.

Having drawn the plan of the observations in Ex
it is required to lay down on it, and to calculate the p
contained in the field-book of the following example.



(2.)

Diagonal. FH 935		
35	560	A
100	320	
88	180	
20	0	
20	695	H
60	513	
	313	
O	300	4
	0	5
	870	G
105	450	
50	0	
4	900	F
98	734	
150	540	
122	330	
40	0	A
At the road.		

Ans. 7·28677 acres.

(3.)

Diagonal. PF 1065		
G	945	5
	878	80
	805	P
44	366	
10	0	
10	950	B
28	825	
90	740	
60	580	
30	430	
30	400	
78	260	
20	0	F
At the road.		

Ans. 7·30361 acr

(4.)

Diagonal. PD 945		
	540	G
	360	58
	260	80
	0	20
70	597	D Γ
98	350	
	0	
	879	C Γ
203	621	
170	421	
	0	P

Ans. 6·50322 acres.

(5.)

Diagonal. EG 670		
4	564	O Γ
70	372	
130	248	
65	100	
12	0	
12	753	E Γ
90	613	
160	518	
170	416	
150	298	
40	0	D

Ans. 4·07145 acres.

The distances not mentioned in these two examples are to be taken from the preceding ones.

Prob. XXIX. To find the contents of a survey.

The areas of single fields, bounded by straight lines, may be found from the lines measured in the field, by the first twelve problems of Mensuration of Superficies.

To calculate Offsets. The most accurate method is to compute them separately, as triangles and trapeziods, by Prob. IV. and VII. of Mensuration of Superficies.

Method 2. If the distances between the perpendiculars be nearly equal. To half the sum of the perpendiculars at the extremities of the base, add all the rest, and multiply the sum by the base, and divide the product by the number of divisions in the base made by these perpendiculars.

Common Method. Divide the sum of the perpendiculars by the number of them for a mean perpendicular, by which multiply the base.

The fourth Example in Prob. XXII. wrought by the first Method.

51 × [illegible]	=	[illegible]	for the	triangle	AKa
[illegible] × [illegible] + [illegible]	=	[illegible]	. . .	trapezoid	KaM
[illegible] × [illegible] + [illegible]	=	[illegible]	. .	trapezoid	[illegible]
[illegible] × [illegible] + [illegible]	=	[illegible]	. .	trapezoid	GdF
[illegible] × [illegible] + [illegible]	=	[illegible]	. . .	trapezoid	FfE
[illegible] × [illegible]	=	[illegible]	. .	triangle	EeD
[illegible] × [illegible]	=	[illegible]	. . .	triangle	ABc
[illegible] × [illegible] + [illegible]	=	[illegible]	. .	trapezoid	BcC
[illegible] × [illegible]	=	[illegible]	. . .	triangle	CeD

2) [illegible]

Ans. [illegible] the whole area.

By the second Method.

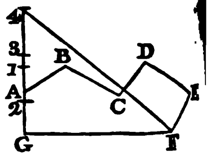
Ans. 725 × ([illegible] × (110 + 135 + 85 + 275 + 185) + (1[illegible] 75) × [illegible]) = [illegible] area.

By the third Method.

Ans. 725 × ([illegible] × (110 + 135 + 85 + 275 + 185) + (1[illegible] 75) × [illegible]) = 196112·5 area.

But surveyors generally endeavour first to obtain a c[illegible] plan of the land, and then they measure, on the plan, [illegible] lines as will enable them to calculate its contents with [illegible] greatest expedition; and for this purpose they reduce [illegible] crooked boundaries to straight lines. Sometimes this is [illegible] by stretching a hair through the crooked part, so that [illegible] small parts cut off by the hair may be equal to the parts ta[illegible] in, as nearly as the eye can judge; and this can be done [illegible] nicely by an experienced surveyor.

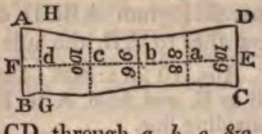
Others reduce the crooked parts to a triangle, [illegible] Prob. XXXIV. of Practical Geometry, which can [illegible] done by the parallel ruler without drawing lines. Th[illegible] suppose ABCDEFG to be the space which is to be reduced to a triangle. Lay the parallel ruler from A to C, and move it till it pass through B, and mark the point 1 in which it cuts AG. Lay the ruler through 1 and D, and move it till it pass through C, and mark 2 where its cuts AG.



Again lay the ruler from 2 through E, and move it till it pass through D, and mark 3 where it cuts AG, and so on; then join 4 and F, and the triangle F4G is equal to the given space. For B1 is parallel to AC; therefore if C1 were drawn, the triangle AC1 = ACB. Now, when the ruler passes through A and C, it takes in the triangle ACB; and when

it is moved to B1, it cuts off the triangle AC1. In like manner the triangle 1D2, which is taken in, is equal to the triangle 1DC cut off; and so of the rest.

Another method of calculation much practised by surveyors is the following, which, though it depend upon judgment, will be found to come very near the truth, and is very expeditious.

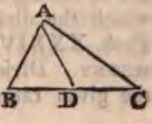


Let ABCD be the plan of a survey, and DC a straight boundary. Draw EF perpendicular to DC, and on it lay a chain, from E to *a*, from *a* to *b*, from *b* to *c*, &c.; and draw parallels to CD through *a*, *b*, *c*, &c., and they will divide the plan into spaces, each a chain in breadth. Measure in a line parallel to DC, half-way between E and *a*. This is supposed to give the mean length of the first space, and therefore is to be measured where the length is a mean, as nearly as the eye can judge. It is here supposed to be 109 links, and is written so in the first space. In the same manner the mean lengths are taken in all the other divisions. After this these lengths are to be added together, and require only three places to be cut off to give the area in acres. The small space ABGH remaining beyond the last parallel, which is only 39 links in breadth, may be found by multiplying 39 by its mean length, judged of as before. Or offsets upon GH may be taken from A and B, and thus a mean breadth may be obtained, to be multiplied by GH, or the mean length. Suppose the offsets at A and B to be 44 and 31, and suppose the mean length to be 96 links; then $96 \times 39 = \cdot 03744$ of an acre. Or the mean offset is $37\cdot 5$, which, multiplied by GH, suppose 100, gives ·03750 of an acre for the content of the part ABGH; and this, added to ·393, the sum of the mean lengths of the other pieces, gives ·4305 of an acre, or 1 rood 28·88 perches, for the whole area.

If the boundary be a curve line, and the distances between the perpendiculars equal, the area may be calculated by Note 2, Prob. XXX. of Mensuration of Superficies.

OF DIVIDING LAND.

Prob. XXVI. To divide a triangular field ABC in any proportion, as that of 9 to 7, by a straight line drawn from the angle A, the opposite side BC being 950 links.



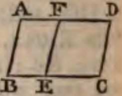
Ans. $16 : 7 :: 950 : 415\frac{5}{8}$ to be laid from B to D; then AD is the dividing line.

2. Divide the triangle ABC, of which the sides are AB 386, BC 428, and AC 533

feet, in the ratio of 8 to 5, by a line drawn from the angle B. Ans. AD 328, and DC 205 feet.

3. Divide the triangle ABC, of which AC is 374, and AB 473 links, and the angle BAC 54°, in the ratio of 5 to 6, by a line drawn from C. Ans. AD 215, and DB 258 links.

PROB. XXVII. To cut off three acres from the parallelogram ABCD of ten acres, by a straight line parallel to AB, the side BC being 495 links.

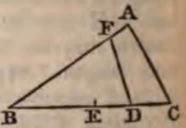


Ans. $10 : 3 :: 495 : 148\frac{1}{2}$ to be laid from B to E, and from A to F; then EF is the dividing line.

2. Divide the parallelogram ABCD, of which AB is 236, and BC 574 yards, and the angle ABC 76°, in the ratio of 3 to 4, by a line parallel to AB. Ans. BE 246, and EC 328 yards.

3. Divide the rectangle ABCD, of which AB is 472, and BC 675 feet, in the ratio of 7 to 8, by a line parallel to AB. Ans. BE 315, and EC 360 feet.

PROB. XXVIII. To cut off two acres from the triangular field ABC of six acres, by a straight line drawn from D, 230 links from B; the line BC being 466 links, and BA 420 links.



Ans. $6 : 2 :: 466 : 155\frac{1}{3} =$ BE, and $230 : 155\frac{1}{3} ::$ BA $420 :$ BF $283\frac{1}{23}\frac{5}{}$; then DF is the dividing line.

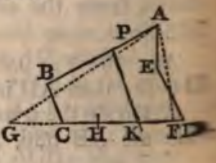
If E had fallen between D and C, then AC must have been divided.

2. Divide the triangle ABC, of which the sides are AB 451, BC 528, and AC 364 links, in the ratio of 7 to 9, by a line drawn from D in BC, 363 links from B. Ans. BF in AB 287 links.

3. Divide the triangle ABC, of which AB is 464, and BC 580 feet, and the angle ABC 64°, in the ratio of 3 to 5, by a line drawn from E in AB, 290 feet from B.

Ans. BF in BC 348 feet from B.

PROB. XXIX. To divide any field ABCDE in a given ratio, as that of 5 to 4, by a straight line drawn from the point P in AB, one of its sides.



Reduce the field to the triangle AFG, having its base in the side CD, which the dividing line will cut, by Prob. XXXIV. of PRACTICAL GEOMETRY. Divide the triangle AFG in the given ratio by the line AH, by

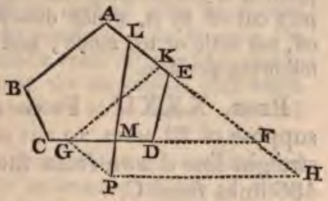
Prob. XXVI. Draw AK parallel to PH, and join PK: it will be the dividing line.

NOTE 1. If the point K fall in CG, the field must be reduced to a triangle which has its base in BC, or a triangle equal to PCK must be made by a line drawn from P to BC.

2. Divide the quadrilateral ABCD, of which the sides are AB 255, BC 284, CD 313, and AD 472 yards, and the angle ABC 57°, in the ratio of 6 to 7, by a line drawn from P in AD, 118 yards from A. Ans. BH 294, and BK 189 yards.

NOTE 2. As the method of dividing the field geometrically by parallels is much easier than the arithmetical, it is best to do it in that way very accurately, and then to measure the result by the scale.

PROB. XXX. To divide a field ABCDE in a given ratio, by a straight line drawn from the point P, without or within the field.

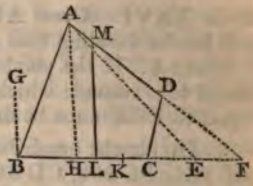


Consider what sides will be cut by the dividing line, suppose AE and CD. Produce these lines till they meet in F, and parallel to them draw PG, PH. Divide the field as in the last problem, by a line GK drawn from G. Find FL a mean proportional between FK and HL, and join PML: it will be the dividing line.

NOTE. To find the point L. Take FH and $\frac{1}{4}$FK, and add them when P is without the triangle, otherwise subtract them, and multiply the sum or difference by FK, and take the square root of the product. The difference between this root and FK will be KL, which is to be laid towards F when P is within the figure, otherwise the contrary way.

2. Divide the pentagon ABCDE, of which the sides are AB 356, BC 381, CD 347, DE 182, and EA 412 feet, and the angles CDE 138°, and AED 124°, into two parts, in the ratio of 2 to 3, by a line drawn from P within the figure; PH parallel to CD being 374, and PG parallel to AE being 38 feet. Ans. FK 440, the root 283, and KL 157 feet.

PROB. XXXI. To divide a field ABCD in a given ratio, by a straight line parallel or perpendicular to a given line, or making a given angle with one of the sides, as BC.



Consider which of the sides are to be cut by the dividing line, as AD and BC. Produce these lines till they meet in F. Reduce the field to a triangle ABE, and divide BE in the given ratio. Draw AH in the position of the dividing line, and make FL equal to the square root of the product of FK and FH, and draw LM parallel to AH, and it will be the dividing line.

2. Divide the quadrilateral figure ABCD, of which AB is 356, BC 528, CD 216, and AD 418 links, and the angle ABC 78°, in the ratio of 3 to 4, by a line perpendicular to BC. Ans. BL 205, or CL 323 links.

These methods of dividing land, though accurate, and in general as short as any, are seldom used by surveyors. They generally draw a line as near as they can conjecture to the position of the dividing line required, and find the area of the part cut off by it, which discovers how much they have cut off, too little or too much; and they alter the line as in the following problem.

Prob. XXXII. From a given field ABCDEF, suppose of 20 acres, to cut off 8 acres towards B, by a straight line drawn from the point G in the line CD, 436 links from C.

Draw GH as near to the position of the line required as you can conjecture, and calculate the area of the space ABCGH, which suppose to be 9·0496 acres: this is 1·0496 acres too much, which, divided by $\frac{1}{2}$GH, suppose 364 links, gives 288 links. Draw a parallel to GH at this distance from it, and let it meet AF in K: then GK is the dividing line required.

When a quantity of land, such as a common, is to be divided among several proprietors in certain proportions, the quantity to be assigned to each will be as the value of his claim, divided by the quality or value of the ground allotted to him. This may be done by adding into two sums the contents and the values: then, by distributive proportion or fellowship, compute the value of each person's share; and from the quality of the ground where his share is to be determined, find what quantity will amount to the value of his share, and lay it off by the last problem.

Suppose it were required to divide 780 acres among three proprietors, whose estates are £1000, £3000, and £4000 a-year, and the values of the land in which their shares are to lie are 5s., 8s., and 10s. the acre respectively.

The claims being as 1, 3, and 4, and the qualities as 5, 8, and 10, the quantities assigned to them must be as $\frac{1}{5}$, $\frac{3}{8}$, and $\frac{4}{10}$, or as 8, 15, and 16, and their shares 160, 300, and 320 acres.

PROB. XXXIII. To transfer, and to enlarge or diminish, a plan.

After the first plan is completed, it will be necessary to draw out a fair one upon vellum or paper.

There are various ways of doing this.

First. If the fields be generally bounded by straight lines, lay the plan upon the clean paper, keeping it firm by weights, and prick through all the corners of the plan, and then connect the points on the clean paper.

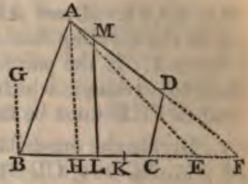
Secondly. Lay a piece of paper covered with black-lead dust between the papers, with the powdered side towards the clean paper, and with a blunt needle trace all the lines on the rough plan with such pressure that the impression may reach the clean paper; after which they are to be traced with ink upon the clean paper.

Thirdly. Divide the rough plan into small squares, and divide the paper to which it is to be transferred into as many squares; then copy the parts of the plan found in each square into the corresponding square of the other plan. In this manner the plan may be enlarged or diminished in any proportion, by making the squares in that proportion.

Fourthly. There are several instruments useful for transferring, enlarging, and diminishing plans, as the proportional compasses, the pentagraph, and the copying-glass.

A plan may be enlarged or diminished in any proportion on the first paper, by Prob. XXXVII. of PRACTICAL GEOMETRY, and afterwards transferred to the clean paper by any of the preceding methods.

After the plan is copied upon the clean paper, write such names, remarks, or explanations as are reckoned to be necessary, and make a fleur-de-lis to point out the direction, and in a convenient corner lay down a scale for measuring the parts of the plan. The title of the plan must be placed in a conspicuous part, and properly ornamented. After which, every part must be coloured or illuminated in the way that appears most natural. Rivers, woods, hills, hedges, houses, roads, &c. must all be distinguished by proper representations. But these things require to be learned by practice.



Consider which of the sides are to be cut by the dividing line, as AD and BC. Produce these lines till they meet in F. Reduce the field to a triangle ABE, and divide BE in the given ratio. Draw AH in the position of the dividing line, and make FL equal to the square root of the product of FK and FH, and draw LM parallel to AH, and it will be the dividing line.

2. Divide the quadrilateral figure ABCD, of which AB is 356, BC 528, CD 216, and AD 418 links, and the angle ABC 78°, in the ratio of 3 to 4, by a line perpendicular to BC. Ans. BL 205, or CL 323 links.

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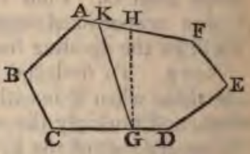


Draw GH as near to the position of the line required as you can conjecture, and calculate the area of the space ABCGH, which suppose to be 9·0496 acres: this is 1·0496 acres too much, which, divided by $\frac{1}{2}$GH, suppose 364 links, gives 288 links. Draw a parallel to GH at this distance from it, and let it meet AF in K: then GK is the dividing line required.

When a quantity of land, such as a common, is to be divided among several proprietors in certain proportions, the quantity to be assigned to each will be as the value of his claim, divided by the quality or value of the ground allotted to him. This may be done by adding into two sums the contents and the values: then, by distributive proportion or fellowship, compute the value of each person's share; and from the quality of the ground where his share is to be determined, find what quantity will amount to the value of his share, and lay it off by the last problem.

DESCRIPTION OF THE SLIDING-RULE.

This rule is 1 foot long, 1·1 inch broad, and ·8 inch thick, and each of its four sides is furnished with a slider.

Upon the first side are four lines, all constructed in the same way, that is, each is divided into 1000 parts, and the numbers are placed at their logarithms. The two on the slider are marked B. or Num. The upper line on the rule is marked A., and the under one M. D. or Malt Depth. This last is inverted, and the point 2218·192 is placed at the right end, so that 10 on this line is opposite to 2218·192 or IM. B. on the line A.

Upon the opposite side of the rule, the lines on the slider are the same with those on the first side. The line on the rule is constructed in the same way, only the distance between 1 and 10 is twice as long. One-half of this line, or from 1 to 3·2, is placed above the slider, and the other half below it. Some rules have a line on which the distance between 1 and 10 is one-third of the distance on this line. On the inside of the slider are the gauge-points for imperial gallons, for imperial malt bushels, and also the multipliers and divisors for square and round vessels.

On the other sides or edges of the rule, the sliders contain lines the same with those of the other sliders; and on the rule are lines for ullaging, on the one edge for ullaging a lying cask, and on the other for a standing cask. These lines are constructed experimentally thus:—Take a cask containing 100 gallons, and fill it with water. Draw off one gallon, and measure the depth of the remaining water; then set the length, or the bung-diameter, according as the cask is standing or lying, on the slider, opposite to 100 on the rule, and opposite to the wet inches on the slider mark 99 on the rule. Draw off another gallon, measure the wet inches, and opposite to them on the slider mark 98 on the rule; and proceed in the same manner till the line is all marked. The inside of the slider, on the edge marked C, contains a line of inches and lines for reducing the first and second varieties of casks to cylinders.

There are several brasses or notches marked on the lines. Thus, on the first side, a brass with IM. B. is marked at 2218·192 for imperial bushels, and another with IM. G. for imperial gallons at 277·274. On the second side are marked on the rule the gauge-points, IM. G. for imperial gallons at 18·789, M. S. or malt bushels in square vessels at 47·097, and M. R. or malt bushels in round vessels at 53·144.

PROB. I. To multiply by the sliding-rule.

Turn up the first side, and set 1 on the slider opposite to the multiplier on the line A; then against the multiplicand on the slider is the product on A.

1. Multiply 15 by 8. Set 1 on B to 8 on A; then opposite to 15 on B will be 120 on A, the product.

NOTE. The 1 at the left end of A may be read 1, or 10, or 100; and the rest of the numbers must be read accordingly, the 2 either 2, or 20, or 200, &c. Also, in reading the multiplicand on the slider, the 1 may be read 10 or 100; but then the product must be increased 10 or 100 times.

2. Multiply 250 by 56. Set 1 on B to 56 on A; then against 250 on B is 14000 on A.

3. Multiply 7·23 by 8·5. . . . Ans. 61·455.
4. . . 82·5 by ·73. . . . . 60·23.
5. . . . ·94 by 7·4. . . . . 6·956.

PROB. II. To divide by the sliding-rule.

Place the divisor on the slider B opposite to the dividend on A; then against 1 on B is the quotient on A.

1. Divide 480 by 15. Set 15 on B to 480 on A; then against 1 on B is the quotient 32 on A.

2. Divide 8142 by 59. . . . Ans. 138.
3. . . 8·75 by 3·25. . . . . 2·69.
4. . . 6·08 by 7·42. . . . ·819.
5. . . 19·7 by 3·5. . . . 5·63.

PROB. III. To work a proportion by the sliding-rule.

Place the first term on the slider B opposite to the second or third on A; then against the other term on B is the answer on A.

1. If 40 yards of cloth cost £24, what will 15 cost?

Ans. Set 40 on B to 15 on A; then against 24 on B is £9 on A, the answer.

2. How many yards of cloth at 18s. may be given for 60 lb. of tea at 7s.? Ans. $23\frac{1}{3}$ yards.

3. If 16 men do a piece of work in 48 days, in what time will 24 men do it? Ans. 32 days.

4. What number of men must be employed to perform in 84 days a piece of work which 108 men perform in 133 days? Ans. 171 men.

5. If £15·6 pay 16 labourers for 18 days, how many, at the same rate, will £35·1 pay for 24 days? Ans. 27 labourers.

6. If 36 yards of cloth, 7 quarters wide, cost £25·2, what will 120 yards of the same quality, 5 quarters wide, cost?

Ans. £60.

PROB. IV. To extract the square root by the sliding-rule.

Take the second side of the rule. Place 1 on the slider opposite to 1 on the rule, then find the given number on the slider, and if it consist of 1, 3, 5, 7, &c. figures, the root is opposite to it on the line above; but if it consist of 2, 4, 6, &c. figures, the root is opposite on the line below it, on the rule.

1. Required the square root of 81. Set 1 on C to 1 on D; then opposite to 81 on C, is 9 on the line below on D.

2. Required the square root of 625. Set 1 on C to 1 on D; then against 625 on C, is 25 on D on the line above.

3. Required the square root of 1681. . Ans. 41.
4. . . . . of 24649. . . 157.
5. . . . . of 5·0625. . . 2·25.
6. . . . . of 30·25. . . 5·5.

PROB. V. To find a mean proportional between two numbers.

Set the lesser on C to the lesser on D; then against the greater on C is the mean proportional on D.

1. Required a mean proportional between 18 and 72.

Set 18 on C to 18 on D; then against 72 on C is 36 on D, which is the mean required.

2. Required a mean proportional between 2448 and 17.

Ans. 204.

3. Required a mean proportional between 128 and 1152.

Ans. 384.

4. Required a mean proportional between 30·25 and 272·25.

Ans. 90·75.

5. Required a mean proportional between 1248 and 78.

Ans. 312.

6. Required a mean proportional between 205·5 and 137.

Ans. 167·79.

PROB. VI. To find a number, which shall have to a given one the same ratio which the squares of two given numbers have to one another.

Set the first term of the ratio on D to the given number on C; then opposite to the other term of the ratio on D stands the answer on C.

1. Required the number which shall be to 36, as the square of 4 to that of 3.

Set 3 on D to 36 on C; then against 4 on D will be 64 on C, the answer.

2. What number is to 120, as the square of 3 to that of 2? Ans. 270.

3. Increase the number 240 in the ratio of the square of 4 to that of 5. Ans. 375.

4. Diminish the number 392 in the ratio of the square of 7 to that of 6. Ans. 288.

5. Find the number to which 196 shall have the same ratio with the square of 7 to that of 9. Ans. 324.

Prob. VII. To find a number which shall be to a given one as the square roots of two given numbers.

Set the first term on C to the given number on D; then against the other term on C stands the answer on D.

1. To what number will 3 have the same ratio with the square root of 108 to that of 48?

Set 3 on D to 108 on C; then against 48 on C is 2 on D, the answer.

2. To what number will 2 be as the square root of 120 to that of 270? Ans. 3.

3. Required the number to which 256 shall be as the square root of 16 to that of 9. Ans. 192.

4. Increase the number 433 in the ratio of the square root of 3 to that of 5. Ans. 559.

5. Diminish the number 1414 in the ratio of the square root of 8 to that of 7. Ans. 1323.

Prob. VIII. Of multipliers, divisors, and gauge-points.

Instead of first finding the content of a vessel in inches, and afterwards reducing it to the measure of capacity required, which must often be done both by multiplying and dividing by known numbers, gaugers find the content in the measure required by means of a single multiplier or divisor.

These multipliers are got by dividing the multiplier used in finding the content by the divisor, which reduces the content to gallons, &c. Thus, to find the multiplier which, in circular vessels, will give the content in imperial gallons, divide ·785398 by 277·274.

To find the divisor which will answer the same purpose, divide 277·274 by ·785398.

Gauge-points are numbers made use of in working by the

sliding-rule. The operation is made similar to that in Prob. VI.; and for that purpose the square root of the divisor is taken for the first term, and is called the Gauge-point.

TABLE I.

MULTIPLIERS, DIVISORS, AND GAUGE-POINTS, FOR CYLINDRICAL VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-Points.
For inches, . .	·7853982	1·273239	1·12838
Imperial gallons, .	·0028326	353·0362	18·7893
Imperial bushels, .	·0003541	2824·2903	53·1441
Green soft soap, lbs.	·0305959	32·6841	5·7170
White soft soap, do.	·0307276	32·5440	5·7047
Cold hard soap, do.	·0289388	34·5557	5·8784
Tallow, gross, do.	·0259379	38·5537	6·2092
Starch, do. . .	·0225689	44·3087	6·6565
Green glass, do.	·0928367	10·7716	3·2820
Plate glass, do. .	·0855740	11·6858	3·4184
Broad glass, do. .	·0746860	13·3894	3·6592
OLD MEASURES.			
Wine gallons, . .	·0034000	294·1183	17·1499
Ale gallons, . .	·0027851	359·0535	18·9487
Corn gallons, . .	·0029219	342·2468	18·4999
Malt bushels, . .	·0003652	2738·0000	52·3259
Scotch pints, . .	·0075372	132·6759	11·5185
Wheat firlots, . .	·0003547	2819·3623	53·0977
Barley firlots, . .	·0002431	4112·9526	64·1323
Irish gallons, . .	·0028955	345·3662	18·5840
Irish barrels, . .	·0000905	11051·7176	105·1296

In this table, the first multiplier is that for finding the area of a circle, and its reciprocal is the first divisor. The other multipliers are got by dividing the first by the number of inches in a gallon, bushel, &c.; and the other divisors by multiplying the first divisor by the number of inches in a gallon, &c. The gauge-points are the square roots of the divisors.

If 1 be put instead of ·7853982 at the top, tables may be formed in the same way for square vessels. Thus, 1 divided by 27·14 gives ·036846, the multiplier for hard soap.

TABLE II.

MULTIPLIERS, DIVISORS, AND GAUGE-POINTS, FOR PRISMATIC VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-Points.
SQUARE.			
Imperial gallons,	·0036065	277·274	16·6516
Imperial bushels,	·0004508	2218·190	47·0977
Hard soap, pounds,	·0368460	27·140	5·2096
Tallow, do. . .	·0330251	30·280	5·5027
Starch, do. . .	·0287356	34·800	5·8992
Green glass, do. .	·1182033	8·460	2·9086
PENTAGONAL.			
Imperial gallons, .	·0062050	161·1610	12·6950
Imperial bushels, .	·0007756	1289·2884	35·9067
Hard soap, pounds,	·0633927	15·7747	3·9717
Tallow, do. . .	·0568190	17·5998	4·1952
Starch, do. . .	·0494390	20·2269	4·4974
Green glass, do. .	·2033661	4·9172	2·2175
HEXAGONAL.			
Imperial gallons, .	·0093700	106·7228	10·3307
Imperial bushels, .	·0011726	853·7824	29·2196
Hard soap, pounds,	·0957287	10·4462	3·2321
Tallow, do. . .	·0858018	11·6548	3·4139
Starch, do. . .	·0746574	13·3945	3·6599
Green glass, do. .	·3071012	3·2563	1·8045
HEPTAGONAL.			
Imperial gallons, .	·0131059	76·3918	8·7351
Imperial bushels, .	·0016382	610·4142	24·7066
Hard soap, pounds,	·1338951	7·4685	2·7329
Tallow, do. . .	·1200103	8·3326	2·8866
Starch, do. . .	·1044228	9·5765	3·0946
Green glass, do. .	·4295405	2·3281	1·5285
OCTAGONAL.			
Imperial gallons, .	·0174139	57·4253	7·5780
Imperial bushels, .	·0021767	459·4027	21·4337
Hard soap, pounds,	·1779081	5·6209	2·3708
Tallow, do. . .	·1594592	6·2712	2·5042
Starch, do. . .	·1387479	7·2073	2·6846
Green glass, do. .	·5707359	1·7521	1·3237

To find the multiplier, divisor, and gauge-point, for imperial gallons in vessels of the form of a regular heptagon.

Divide the tabular multiplier 3·6339124 by 277·274: the quotient ·0131059 will be the multiplier. Divide 277·274 by 3·6339124: the quotient 76·3918 will be the divisor, and its square root 8·7351 will be the gauge-point.

In the same manner the multipliers, divisors, and gauge-points are found for any regular polygon.

TABLE III.

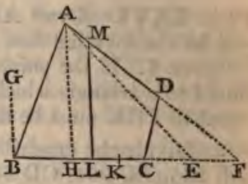
MULTIPLIERS, DIVISORS, AND GAUGE-POINTS, FOR CONICAL VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-Points.
For inches, . .	·2617994	3·819717	1·59441
Imperial gallons, .	·0009442	1059·1086	32·5441
Imperial bushels, .	·0001180	8472·8708	92·0490
Soft soap, pounds,	·0101986	98·0522	9·9021
White soft soap, do.	·0102425	97·6320	9·8810
Hard soap, do. .	·0096463	103·6671	10·1817
Tallow, do. . .	·0086460	115·6611	10·7547
Starch, do. . .	·0075230	132·9261	11·5295
Green glass, do. .	·0309456	32·3148	5·6846
Plate glass, do. .	·0285247	35·0575	5·9208
Broad glass, do. .	·0248953	40·1683	6·3379
OLD MEASURES.			
Wine gallons, . .	·0011333	882·3549	29·7045
Ale gallons, . .	·0009284	1077·1605	32·8201
Malt bushels, . .	·0001217	8214·0000	90·6306
Scotch pints, . .	·0025124	398·0277	19·9506
Wheat firlots, . .	·0001182	8458·0870	91·9680
Barley firlots, .	·0000810	12338·8578	111·0812

In pyramidal, conical, &c. vessels, where, in finding the content, we multiply by one-third of the length, the multiplier should be one-third of that in the table, the divisor must be three times as large as that in the table, and the gauge-point must be the square root of three times the tabular divisor; and, in this case, use the whole length, instead of one-third of it. The same remarks are applicable to rules in which we multiply by any other part of the length.

PROB. IX. To gauge areas one inch deep.

I. When one side is given, set the gauge-point on D to 1 on C; and against the given side on D is the answer on C.

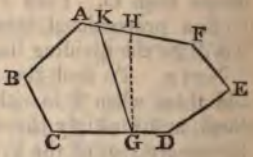


Consider which of the sides are to be cut by the dividing line, as AD and BC. Produce these lines till they meet in F. Reduce the field to a triangle ABE, and divide BE in the given ratio. Draw AH in the position of the dividing line, and make FL equal to the square root of the product of FK and FH, and draw LM parallel to AH, and it will be the dividing line.

2. Divide the quadrilateral figure ABCD, of which AB is 356, BC 528, CD 216, and AD 418 links, and the angle ABC 78°, in the ratio of 3 to 4, by a line perpendicular to BC. Ans. BL 205, or CL 323 links.

These methods of dividing land, though accurate, and in general as short as any, are seldom used by surveyors. They generally draw a line as near as they can conjecture to the position of the dividing line required, and find the area of the part cut off by it, which discovers how much they have cut off, too little or too much; and they alter the line as in the following problem.

PROB. XXXII. From a given field ABCDEF, suppose of 20 acres, to cut off 8 acres towards B, by a straight line drawn from the point G in the line CD, 436 links from C.



Draw GH as near to the position of the line required as you can conjecture, and calculate the area of the space ABCGH, which suppose to be 9·0496 acres: this is 1·0496 acres too much, which, divided by $\frac{1}{2}$GH, suppose 364 links, gives 288 links. Draw a parallel to GH at this distance from it, and let it meet AF in K: then GK is the dividing line required.

When a quantity of land, such as a common, is to be divided among several proprietors in certain proportions, the quantity to be assigned to each will be as the value of his claim, divided by the quality or value of the ground allotted to him. This may be done by adding into two sums the contents and the values: then, by distributive proportion or fellowship, compute the value of each person's share; and from the quality of the ground where his share is to be determined, find what quantity will amount to the value of his share, and lay it off by the last problem.

Suppose it were required to divide 780 acres among three proprietors, whose estates are £1000, £3000, and £4000 a-year, and the values of the land in which their shares are to lie are 5s., 8s., and 10s. the acre respectively.

The claims being as 1, 3, and 4, and the qualities as 5, 8, and 10, the quantities assigned to them must be as $\frac{1}{5}$, $\frac{3}{8}$, and $\frac{4}{10}$, or as 8, 15, and 16, and their shares 160, 300, and 320 acres.

PROB. XXXIII. To transfer, and to enlarge or diminish, a plan.

After the first plan is completed, it will be necessary to draw out a fair one upon vellum or paper.

There are various ways of doing this.

First. If the fields be generally bounded by straight lines, lay the plan upon the clean paper, keeping it firm by weights, and prick through all the corners of the plan, and then connect the points on the clean paper.

Secondly. Lay a piece of paper covered with black-lead dust between the papers, with the powdered side towards the clean paper, and with a blunt needle trace all the lines on the rough plan with such pressure that the impression may reach the clean paper; after which they are to be traced with ink upon the clean paper.

Thirdly. Divide the rough plan into small squares, and divide the paper to which it is to be transferred into as many squares; then copy the parts of the plan found in each square into the corresponding square of the other plan. In this manner the plan may be enlarged or diminished in any proportion, by making the squares in that proportion.

Fourthly. There are several instruments useful for transferring, enlarging, and diminishing plans, as the proportional compasses, the pentagraph, and the copying-glass.

A plan may be enlarged or diminished in any proportion on the first paper, by Prob. XXXVII. of PRACTICAL GEOMETRY, and afterwards transferred to the clean paper by any of the preceding methods.

After the plan is copied upon the clean paper, write such names, remarks, or explanations as are reckoned to be necessary, and make a fleur-de-lis to point out the direction, and in a convenient corner lay down a scale for measuring the parts of the plan. The title of the plan must be placed in a conspicuous part, and properly ornamented. After which, every part must be coloured or illuminated in the way that appears most natural. Rivers, woods, hills, hedges, houses, roads, &c. must all be distinguished by proper representations. But these things require to be learned by practice.

7. Required the content, in imperial bushels, of a regular hexagon, of which the side is 138 inches.

Ans. 22·331 bushels.

III. If the inches in any of the gauge-points be laid on a rule, and this distance be divided into 100 equal parts, the dimensions may be taken with that rule, and then the content may be found without using the multipliers or divisors. Thus, if 7·64 inches be divided into 100 equal parts, the side of the octagon in Ex. 6, measured by this rule, would be 5·496 ale gallons; and this, multiplied by itself, would give 30·206 ale gallons for the content.

1. Required the content, in imperial gallons, of a circle, of which the diameter is 40 inches.

Ans. $1600 \times {\cdot}0028326 = 4{\cdot}52316$ imperial gallons.

By the Gauge-Points.

Set 18·8 on D to 1 on C; then against 40 on D is 4·53 gallons on C.

If 18·8 inches be divided into 100 equal parts, the diameter measured by this scale would be 2·13, which, multiplied by itself, gives 4·5369 imperial gallons for the content.

2. Required the content, in imperial gallons, of a sector of a circle, of which the radius is 42 inches, and the arc 118 inches. Ans. 8·9369 imperial gallons.

3. Required the content, in hard soap, of a trapeze, the diagonal being 32 inches, and the perpendiculars upon it from the angles 18 and 14 inches. Ans. 18·865 lbs.

4. Required the content of a quadrant, at 1 inch deep, in plate glass, the radius being 16 inches. Ans. 21·907 lbs.

IV. Some preparation is often necessary before the question can be wrought by the sliding-rule, as in the following examples.

1. Required the content, in imperial gallons, of a segment of a circle, the diameter 50, and the versed sine 10 inches.

Ans. $10{\cdot}000 \div 50 = {\cdot}200$ the tabular versed sine, opposite to which is ·1118238 the tabular area; and ${\cdot}1118238 \times 50^2 \times {\cdot}0036065 = 1{\cdot}00823$ imperial gallon.

Set 16·65 on D to 1118 on C; and at 50 on D is 1·01 imperial gallon on C.

2. Required the content, at 1 inch deep, in tallow, of a triangular vessel, of which the sides are 36, 24, and 20 inches.

Here the half sum is 40, and the remainders are 20, 16, and 4. A mean proportional between 20 and 40 is 28·284, and between 16 and 4 is 8. Then,

Set 30·28 on A to 28·284 on B; and against 8 on A is 7·47 lbs. tallow on B.

3. What is the content, in imperial gallons, of an ellipse, of which the axes are 72 and 50 inches?

Ans. 10·17936 imperial gallons.

Prob. X. To gauge solids.

When the depth is greater than one inch, set the gauge-point to the depth instead of 1.

1. Required the content, in imperial gallons, of a rectangular prism, of which the length is 81, the breadth 26, and the depth 25 inches.

Ans. $81 \times 26 \times 25 \times \cdot 0036065 = 189 \cdot 882$ imp. gallons.

By the Gauge-Points.

Set 25 on C to 25 on D; and at 81 on C is 45 on D, a mean proportional between 25 and 81. Then,

Set 16·65 on D to 26 on C; and at 45 on D is 190 imperial gallons on C.

2. Required the content, in imperial gallons and bushels, of an octagonal prism, of which the depth is 80 inches, and each side of the base 63 inches.

Ans. $63^2 \times 80 \times \cdot 0174139 = 5529 \cdot 262$ imperial gallons, $=$ 691·158 bushels.

By the Sliding-Rule.

Set 7·578 on D to 80 on C; and at 63 on D is 5529·3 imperial gallons on C.

Set 21·434 on D to 80 on C; and at 63 on D is 691·16 imperial bushels on C.

3. Required the content, in imperial gallons, of a cylindrical vessel, the depth 40 inches, and the diameter of the base 27 inches.

Ans. $27^2 \times 40 \times \cdot 0028326 = 82 \cdot 599$ imperial gallons.

By the Sliding-Rule.

Set 18·8 on D to 40 on C; and at 27 on D is 82·6 imperial gallons on C.

4. Required the content, in imperial gallons, of the frustum of a square pyramid, the depth 24 inches, each side of the lower base 26, and of the higher 34 inches.

Ans. $34 + 26 = 60$, and $(60^2 - 34 \times 26) \times 24 \times \cdot 0012022$ $= 78 \cdot 364$ imperial gallons.

By the Sliding-Rule.

First set 26 on C to 26 on D; and at 34 on C is 29·72 on D, the mean proportional between 26 and 34. Then,

Set 28·84 on D to 24 on C; and at 60·0 on D is 104·2 on C.
. . 28·84 . . 24 . . . . . 29·7 . . —25·8 . .

78·4 im. gal.

NOTE. These, with most other questions, may be wrought more easily by Prob. XII. of MENSURATION OF SOLIDS; and therefore it is proper to give tables for it, and the rule for working it by the sliding-rule.

TABLE IV.

GAUGE-POINTS TO BE USED WHEN THE MIDDLE AREA IS TAKEN.

Measures.	Gauge-Points.	
	For Squares.	For Circles.
For inches,	1·	2·764
Imperial gallons,	40·7878	46·024
Imperial bushels,	115·3653	130·176
Soft soap, pounds,	12·4105	14·004
White soft soap, do.	12·3839	13·974
Hard soap, do.	12·7609	14·399
Tallow, do.	13·4789	15·209
Starch, do.	14·4499	16·305
Green glass, do.	7·1246	8·039
Plate glass, do.	7·4208	8·373
Broad glass, do.	7·9433	8·963
OLD MEASURES.		
Wine gallons,	37·2290	42·008
Ale gallons,	41·1339	46·415
Corn gallons,	40·1597	45·315
Malt bushels,	113·5892	128·172
Scotch pints,	25·0044	28·214
Wheat firlots,	115·2646	130·062
Barley firlots,	139·2187	157·091
Irish gallons,	40·3423	45·522
Irish barrels,	228·2104	257·508

Rule for working by the Pen.

Find the squares or products of the sides or diameters at the top and bottom, and of the double of those in the middle: the sixth part of the sum of these, multiplied by the proper multiplier in Table I. or II. will give the content.

By the Sliding-Rule.

Set the gauge-point on D to the length on C; then opposite to the sides or diameters at the ends, and to twice that in the middle on D, will be found three numbers on C; and these three, added together, will give the content.

To work the last question by this rule. $\frac{1}{6}(26^2+34^2+60^2) \times 24 \times \cdot0036065 = 78\cdot362$ imperial gallons.

By the Sliding-Rule.

Set 40·79 on D to 24 on C; then at 60 on D is 51·6 on C.
. . 40·79 . . 24 . . . . . 34 . . 16·6 . .
. . 40·79 . . 24 . . . . . 26 . . 9·65 . .

77·85 imp. gal.

5. Required the content, in imperial gallons, of a frustum of a rectangular pyramid, the depth of the frustum 100 inches, the sides of the upper base 18 and 8 inches, and the sides of the lower base 27 and 12 inches.

Ans. $\frac{1}{6}(18 \times 8 + 27 \times 12 + 45 \times 20) \times 100 \times \cdot0036065 = 82\cdot2282$ imperial gallons.

By the Sliding-Rule.

Set 40·79 on D to 100 on C; then at 18 on D is 19·47 on C.
. . 40·79 . . 100 . . . . . 12 . . 8·65 . .
. . 40·79 . . 100 . . . . . 30 . . 54·09 . .

82·21 imp. gal.

6. Required the content, in imperial gallons, of the frustum of a cone, the depth of the frustum 100 inches, and the diameters of the bases 18 and 12 inches.

Ans. $\frac{1}{6}(18^2 + 12^2 + 30^2) \times 100 \times \cdot0028326 = 64\cdot58328$ imperial gallons.

Set 46·02 on D to 100 on C; then at 18 on D is 15·3 on C.
. . 46·02 . . 100 . . . . . 12 . . 6·8 . .
. . 46·02 . . 100 . . . . . 30 . . 42·5 . .

64·6 imp. gal.

7. If the axis of a globe be 100 inches, how many imperial gallons will it contain?

In a sphere, the square of twice the middle diameter is three times the square of the axis.

Ans. $\frac{1}{6}(10000 + 30000 + 0) \times 100 \times \cdot0028326 = 1888\cdot4$ imperial gallons.

Set 46·02 on D to 100 on C; then at 200 on D is 1888·7 imperial gallons on C.

8. Required the content, in imperial gallons, of a bowl or

subtend the same angle BDC. After finding the angle at B, work the triangle DBC.

NOTE 3. If the three places A, B, C, be in a straight line, the first operation will not be required. The rest are the same as before.

3. The three sides of the triangle ABC are AB 280, BC 314, and AC 326 yards; and from the station D without the triangle, the angle ADB was 25° 52′, and ADC 23° 6′, the point C being the nearest to D. Required their distances from D. Ans. AD 586·154, BD 413·41, CD 308·107 yards.

4. Suppose AB 267 feet, BC 209, and AC 346, and at the point D, within the triangle, the angle ADC is 128° 40′, and ADB 91° 20′. Required the distances of D from the angles.

Ans. AD 104·05, BD 189·33, and DC 178·85 feet.

NOTE. When D is in one of the sides, describe a segment on BC containing the given angle.

5. Suppose AB 122·4, BC 74, and AC 82 chains, and at D in AB, produced beyond B, the angle ADC is 22° 45′. Required the distance of D from the angles.

Ans. AD 181·8, BD 59·4, and CD 125·4 chains.

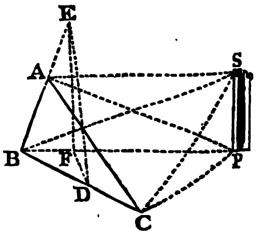
6. Suppose AB 1234, BC 873, and AC 632 yards, and at D in AB the angle ADC is 120°. Required its distance from the angles.

Ans. AD 226·12, BD 1007·88, and CD 487·84 yards.

7. Suppose AB 138, BC 224, and AC 326, and at D the angles are ADB 7° 22′, and ADC 19° 58′. Required the distance of D from the angles.

Ans. AD 510·96, BD 385·286, and DC 204·87.

PROB. XV. Given the angles of elevation of a tower PS, taken at three stations A, B, and C, on a level plane, no two of which are in the same vertical plane with the tower, viz. PAS 20° 10′, PBS 18° 50′, and PCS 34° 30′, and also the distances between the stations AB 324, BC 568, and AC 672 yards; to find the height of the tower.



Make the triangle ABC, of which AB is 324, BC 568, and AC 672, and make BE = BC, and BD = BA, and join ED, and upon it make the triangle EDF on either side of DE, so that BE : EF : : cot. PBS : cot. PAS, and BD : DF : : cot. PBS : cot. PCS; or make EF 527·494, and DF 160·79, and join BF, and

make the angle $BAP = BFE$. Then erect PS perpendicular to the plane ABP, and in the plane passing through AP and PS make the angle PAS $20^\circ\ 10'$, and PS will be the tower required.

Join PC, CS, BS, the triangles APB, FBE, being similar, $AP : PB :: FE : EB ::$ cot. SAP : cot. SBP, therefore SBP is $18^\circ\ 50'$; also $PB : BE = BC :: BA = BD : BF$, therefore the triangles PBC and FBD are similar; and $BP : PC :: BD : DF ::$ cot. PBS : cot. PCS, therefore PCS is $34^\circ\ 30'$.

In each of the triangles EBD, EFD, are given the three sides, to find the angles BED $28^\circ\ 45'\ 30''$, and FED $6^\circ\ 47'\ 26''$; and their difference $21^\circ\ 58'\ 4''$, or their sum $35^\circ\ 32'\ 56''$, is the angle BEF, from which, with the sides BE and EF, the angle BFE or BAP is found in the first case to be $89^\circ\ 48'\ 7''$, and in the other $78^\circ\ 48'\ 22''$. Therefore AP is 866·108 or 546·676, and PS 318·094 or 200·78.

2. Let AB be 326, BC 584, and AC 683, and the angles of elevation SAP 30°, SBP 26°, and SCP 23°; to find PS.

Ans. PS is 952·14 or 168·642.

3. Let AB be 80, BC 119, and AC 140 yards, and the elevation at A 50°, at B 60°, and at C 55°. Required the height of the object D. Ans. 96·4 feet.

4. Let AB be 60, BC 72, and AC 132 feet, and the elevations of S at A $30^\circ\ 48'$, at B $40^\circ\ 33'$, and at C $50^\circ\ 23'$. Required the height of S. Ans. 94·84 feet.

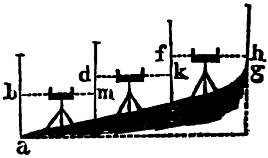
5. Let AB and BC be each 84 feet, and the points A, B, C, in a straight line, and the elevation at A $36^\circ\ 50'$, at B $21^\circ\ 24'$, and at C 14°. Required the height of the object.

Ans. 53·96 feet.

OF LEVELLING.

When the altitudes of the several parts of an irregular ascent are to be determined, a spirit-level with telescopic sights is to be used.

Prob. XVI. To find the height of *g* above *a*.



Erect a pole *ab* at *a*, and another *cd* at a convenient distance. Place the level between them, and, directing the sights to the pole *ab*, cause the point *b* to be marked on it; then direct the sights to the pole *cd*, and on it mark *m*. Next erect a pole nearer to *g*, as at *e*, and place the level between it and the pole *cd*, and mark upon them, as before, the points *d* and *k*; and proceed in this way to g.

To find the height of g above a, take the sum of the heights ab, cd, &c. got by looking towards a, and from it subtract the sum of the heights cm, ek, &c. got by looking towards g: the remainder is the height of g above a. In like manner the heights of c, e, &c. above a are got. If the horizontal distance between a and g be required, add bm, dk, &c.

To find the height of any point c in a regular ascent: The distance ag is to ac, as the height of g above a to the height which c ought to have above a.

It is not necessary to place the poles in the same direction with ab and gh, but it is necessary to erect them perpendicular, or nearly so.

NOTE. When the distance between the poles ab and cd is very great, the line bm will differ a little from the true level; for bm is a tangent to a great circle of the earth, passing through the centre of the instrument, and the true level is the arc of that circle between the poles ab and cd. The correction may in general be neglected: for a mile it is 7·96 or 8 inches; and for other distances from the instrument, the correction varies as the square of the distance.

1. Let the heights on the poles taken by looking down the eminence be 11, 8, 5, 6, 4, and those taken by looking up be 5, 3, 1, 4, 6 feet. Required the height of the eminence.

Ans. 15 feet high.

2. Let the heights taken by looking down be 10, 11, 7, 5, 8, 4, 9, and those taken by looking up be 3, 5, 2, 6, 4, $5\frac{1}{2}$, $3\frac{1}{2}$ feet. Required the height of the eminence. And, supposing the sloping distance from the bottom to the top to be 346 feet. Required the height in a regular slope at the distance of 136 feet from the bottom.

Ans. 25 feet high in all, and, at 136 feet, 9·8266 feet.

3. A hollow in a road, of which the depth on the lowest side is 56 feet, and on the upper 74, and the width at the top of the lower side is 234 feet, and at the bottom 87, and halfway up 172 feet, is to be filled up from the road on the upper bank, so as to form a regular slope. How much of the road must be excavated?

Ans. 1263·18 feet.

TO MEASURE HEIGHTS BY THE BAROMETER.

The elasticity or the density of the air is as the weight of the superincumbent atmosphere; and therefore, if the heights vary in arithmetical progression, the densities will vary in geometrical progression; that is, the height is as the logarithm of the density. It has been found by experiment, that the module of the barometrical logarithms is 10,000 times that of the common logarithms; wherefore, if B be the height of the

mercury at the lower station, and b that at the higher, and h the difference of the heights of the stations, then $h = 10{,}000 \times (\text{com. log. B} - \text{com. log. } b)$ expressed in fathoms. But this formula is true only upon the supposition that the temperature of the air is 32°, and that it is the same at both stations; neither of which is exactly true.

It is found by experiment, that quicksilver expands about $\frac{1}{10000}$ part of its bulk for every degree of Fahrenheit's thermometer. Let r be the temperature at the lower station, and r' that at the higher, as indicated by the thermometer attached to the barometer, then $b + \frac{r - r'}{10000} b$ will be the height of the mercury at the higher station, when reduced to the same temperature with that at the lower station; and thus $h = 10000 \times \left(\log. \text{B} - \log. \left(b + \frac{r - r'}{10000} b\right)\right)$.

Again, the air expands nearly ·00245 of its bulk for every degree of Fahrenheit's thermometer. Let t be the temperature of the air at the lower station, and t' that at the higher, as indicated by a thermometer in the open air, then $\frac{1}{2}(t + t')$ may be taken for the mean temperature; and therefore the former formula has to be multiplied by $·00245 \times \left(\frac{t + t'}{2} - 32\right)$ for an additional correction.

PROB. XVII. To find the height of one place above another.

From what has been shown, the complete formula will be $h = 10000 \times \left(\log. \text{B} - \log. \left(b + \frac{r - r'}{10000} b\right)\right) \times \left(1 + ·00245 \times \left(\frac{t + t'}{2} - 32\right)\right)$, which, expressed in words, gives the following

RULE. Divide the difference of the heights of the attached thermometer by 10000, and add 1 to the quotient, and add the logarithm of the sum to the logarithm of the height of the barometer at the highest station, and subtract the sum from the logarithm of the height of the barometer at the lower station: the remainder, multiplied by 10000, will give the approximate height. Take the difference between 32° and half the sum of the heights of the detached thermometer, and multiply it by ·00245; and if the half sum of the heights be greater than 32°, add the product to 1, otherwise subtract; and the sum or remainder, multiplied by the approximate height, will give the true height.

Note. This method of finding heights is convenient, but it is not very accurate.

1. Suppose the height of the mercury in the barometer at the bottom of the hill to be 29·56 inches, and at the top 28·27 inches, and the temperature of the mercury 63° and 54°, and the temperature of the air 56° and 48°. Required the height of the hill.

Ans. $\frac{63-54}{10000}=\cdot0009$ and $10000\times(\log.\ 29\cdot56-\log.\ 28\cdot27-\log.\ 1\cdot0009)=10000\times(1\cdot4707044-1\cdot4513258-0\cdot0003907)=10000\times\cdot0189879=189\cdot879$ fathoms $=$ 1139·274 feet, the approximate height. Also, $\frac{1}{2}(56+48)-32=20$, and $1+20\times\cdot00245=1\cdot0489$; therefore $1139\cdot274\times1\cdot0489=1195\cdot098$ feet, the true height.

2. Let the height of the barometer at the lower station be 29·57, and at the higher 28·7 inches, the height of the attached thermometer at the lower 55·28°, and at the higher 51·75°, and the temperature of the air at the lower 54°, and at the higher 50·5°. Required the elevation. Ans. 807·117 feet.

3. Let the heights of the barometer be 29·4 and 25·19 inches, the attached thermometer 50° and 46°, and the temperature of the air 45° and 39°. Required the elevation.
Ans. 686·458 fathoms.

4. Let the heights of the barometer be 29·89 and 26·27 inches, the attached thermometer 56·5° and 42·75°, and the temperature of the air 55·25° and 43°. Required the elevation. Ans. 3467·783 feet.

Prob. XVIII. To measure distances by sound.

Rule. Multiply the time the sound takes in seconds by 1142: the product will be the distance in feet.

Note. Sound in common air moves uniformly at the rate of 1142 feet in a second. Cold, and uneven surfaces, retard its motion a little, and heat accelerates it in a small degree.

1. I observed the flash of a gun 30 seconds before I heard the report. How far was it distant from me?
Ans. $30\times1142=34260$ feet.

2. I observed a flash of lightning, and after 6 strokes of my pulse I heard the thunder, and my pulse makes 68 strokes in a minute. How far was the thunder distant from me?
Ans. 1 mile 255 yards.

3. How long, after firing a gun, will it be till the report is heard at the distance of 8 miles? Ans. 37 seconds.

4. A person standing on the bank of a river heard the echo of his voice reflected from a rock on the opposite bank, in 4

seconds after he uttered it. What is the breadth of the river? Ans. 2284 feet.

Prob. XIX. To measure a height by the descent of a stone, &c.

Rule. Multiply the square of the time of descent in seconds by $16\frac{1}{12}$: the product will be the height in feet.

To find the time of descending. Divide the height in feet by $16\frac{1}{12}$, and the square root of the quotient will be the time in seconds.

Note. A heavy body descends $16\frac{1}{12}$ feet in the first second of time, and the spaces descended are as the squares of the times.

1. A stone takes 3 seconds in falling from the top of a tower to the ground. What is the height of the tower?
Ans. $3 \times 3 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

2. In what time will a stone dropt from the height of 579 feet reach the ground? Ans. 6 seconds.

3. What is the height of a precipice, when a stone takes 7 seconds in falling from the top to the bottom?
Ans. $788\frac{1}{12}$ feet.

4. I reckoned 7 strokes of my pulse during the falling of a stone from the top of a rock. What height did it fall, the pulse beating 70 times in a minute? Ans. 579 feet.

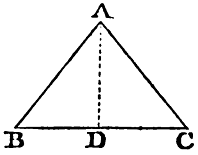
5. While a stone descended from the top of a tower, a pendulum 10 inches long made 8 vibrations. Required the height.
Ans. 263 feet.

TO SURVEY FIELDS.

Prob. XX. To survey a triangular field ABC.

First, with the Chain only. Measure the three sides by Prob. I.

Secondly, with the Chain and Cross. Measure along BC by Prob. I., and with the cross find the point D, where the perpendicular from A meets BC, by Prob. VI. Write down the measures of BD, BC, and DA.



Thirdly, with the Theodolite and Chain. Measure one angle ABC by Prob. III., and the containing sides AB and BC by Prob. I. Or measure BC by Prob. I., and two angles ABC and ACB by Prob. III. From these measures the plan may be easily drawn by Prob. XIX. XX. or XXI. of Practical Geometry; and the area may be found by Prob. IV. V. or VI. of Mensuration.

1. In a triangular field I measured the base 856 linl found the extremity to be the foot of the perpendicula it, which I measured 672 links. Required the content.

Ans. 2 acres 3 roods 20 perches 5 yards 5·53 squa

2. In measuring the base of a triangular field, I fou foot of the perpendicular 256 links from its extremit base 927 links, and the perpendicular 582 links. Re the area. Ans. 2 acres 2 roods 31 perches 18 yards 4·

3. I measured an angle of a triangular field 73° 24′ the sides containing it 688 and 492 links. Required th of the field, and the area.

Ans. 1 acre 2 roods 19 perches 15 yards

4. I measured one side of a triangular field 1268 lin took the angles at its extremities 57° 36′ and 62° 24′— quired the area.

Rule. Add the sines of the given angles and the log. side, and subtract the sine of the third angle, or of the su the given ones, to get the perpendicular = 1095·55.

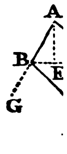
Ans. 6 acres 3 roods 31 perches 9·8591 y

5. The three sides of a triangular field are 1275, 987, 642 links. Required the area.

Ans. 3 acres 17 perches 24 yards 3·106

Prob. XXI. To survey a field contained by sides.

Fig



First, with the Chain only. Measure the four sides and a diagonal BD by Prob. I.

Secondly, with the Chain and Cross. Measure along a diagonal BD by Prob. I., and, with the cross, find by Prob. VI. the points E and F, upon which the perpendiculars fall from A and C, and write the lengths of BE, BF, BD, AE, and CF.

Fi

Or measure the longest side BC, marking E and F the places of the perpendiculars, and measure AE and DF.

Thirdly, with the Theodolite and the Chain. Place the theodolite at B (fig. 1,) and take the angles ABD and DBC by Prob. III., and measure the diagonal BD by Prob. I., and again at D tal angles ADB and BDC. Or take the angle ABC, and m the four sides.

If the angle ABC cannot be measured conveniently the field, fix a pole G in the direction of either sid extended beyond B, and measure the angle CBG, subtracted from 180°, will give ABC.

Fourthly, with the Plane-Table and the Chain. Place the le at one of the angles B, from which all the other angles y be seen, and turn it round till the needle points to the r-de-lis, and there fix it. Fix also a pin in some part of paper to represent B. Apply the fiducial side of the x to the pin, and turn it till the angle A is seen through sights. Draw a line from the pin in that direction. sure BA, and by the scale on the index lay it on that from B to A. Next turn the index till the angle D is through the sights, and draw a line in that direction, and t lay the length of BD. Lastly, draw a line in the direc- of C, and on it lay BC, and join CD and DA. In the e manner any field may be surveyed by the plane-table, an angle can be taken, from which all the other angles he field are seen.

1. I measured along the diagonal BD, (fig. 1,) and at E, 18 links from B, was the foot of the perpendicular AE 318, nd at F, 527 links from B, was the foot of the perpendicular F on the opposite side of BD, 426 links: the whole length f the diagonal BD was 968 links. Required the plan and he area.

Ans. Area 3 acres 2 roods 16 perches 4 yards 5·8176 feet.

2. I measured along BC the longest side of a four-sided ld ABCD, (fig. 2,) and at E, 125 links from B, was the ot of the perpendicular AE, which measured 624 links, d at F, 635 from B, was the foot of another perpendicular D 462 links: the whole length of the side BC was 1274 nks. Required the plan and the area.

Ans. Area 4 acres 2 roods 21 perches 20·0376 yards.

3. I measured an angle ABC of a quadrilateral field 128°, d the four sides AB 536 links, BC 843, CD 634, and AD 96 links. Required the plan and the area.

Ans. Area 4 acres 2 roods 26 perches 16 yards $5\frac{1}{4}$ feet.

4. I measured the diagonal BD of a four-sided field 1462 nks; and at its extremities I took the angles which it made ith the sides, viz. ABD 48° 20′, CBD 41° 26′, ADB 29° 0′, and BDC 38° 44′. Required the plan and the area.

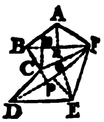
Ans. 8 acres 2 roods 4 perches 28 yards $3\frac{1}{4}$ feet.

5. In taking the plan of a quadrilateral field by the plane- ble, I found the straight side AB to lie N. 73° E., and to easure 568 links; the diagonal AC to lie S. 83° E., 978 nks; and the side AD to lie S. 47° E., 734 links. Required e plan and the area.

Ans. 3 acres 38 perches 9 yards 3·071 feet.

Prob. XXII. To survey any field with the chain.

First, with the Chain only. Measure all the sides of the field, and then the diagonals BF, FC, FD. From these the field may be drawn upon paper by Prob. XXVIII. of Practical Geometry, and its area may be found by Prob. XI. of Mensuration of Superficies.



1. In a six-sided field I measured all the sides, viz. AB 583 links, BC 324, CD 456, DE 892, EF 728, and AF 477 links, and from F measured the diagonals FB 897, FC 723, and FD 948 links. Required the plan and the area.

Ans. 7 acres 12·9 yards.

Secondly, with the Chain and Cross. Divide the field by diagonals into as many trapezes as possible, and the remainder will consist of one or more triangles. Thus the field ABCDEF may be divided into two trapezes ABCF and CDEF, by joining CF. These may be surveyed as in the last Problem.

2. In a heptagonal field I measured along the northernmost diagonal BG, and at 207 links from B found the foot of a perpendicular above it AH, which measured 272; and at 578 from B found the foot of a perpendicular under it FK, which measured 498; the diagonal BG 928. From F, I measured along a diagonal FC, and at 488 from F was at the foot of the perpendicular from B, which measured 587, and the diagonal FC 896. Then, from C, I measured along a diagonal CE, and at 498 from C was the foot of an under perpendicular ND 630, and at 688 from C was at the foot of a perpendicular FM 574 links; the diagonal CE was 1093 links. Required the plan and the area.

Ans. 12 acres 3 roods 5 perches 5 yards 5·965 feet.

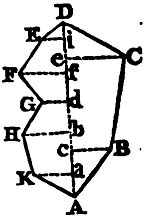
Note. If a perpendicular, as E*p*, upon a diagonal DF, fall without the field, and it be inconvenient to measure it in that situation, the other diagonal CE, with the perpendiculars upon it, may be taken; or the two triangles DEF, CDF, may be measured separately.

3. In a hexagonal field ABCDEF, I measured along the diagonal BF, and, at 328 links from B, I was at the foot of the perpendicular AG, which measured 286, and the diagonal BF was 536; but had to measure 127 links farther without the field, to come to the foot of the perpendicular EH on the opposite side of BF, which measured 453. Again, measuring along the diagonal EC, I found, at 386 from E, the foot of the perpendicular DK, which measured 496; and, 674 from E, found the foot of the perpendicular BL, which mea-

sured 486; the whole length of the diagonal EC was 895 links. Required the plan and the area.

Ans. 6 acres 24 perches 5 yards 8·1432 feet.

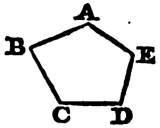
Thirdly. In fields not very large, it will be sufficient to measure one diagonal, and the perpendiculars upon it from all the other angles.



4. Suppose the distances of the perpendiculars from A to be 50, 145, 220, 295, 380, 475, and 655, the whole line AD being 725 links, the second and sixth distances reach to perpendiculars on the right hand, and the rest to those on the left hand. Also the perpendiculars on the right are 75 and 150, and the others in their order are 110, 135, 85, 275, and 185 links. Required the plan and the area.

Erect perpendiculars upon AD, at their proper distances from A; and, having made them of their proper length, the plan is drawn by joining their extremities. The area is easily found by Prob. IV. and VII. of MENSURATION OF SUPERFICIES. Ans. 1 acre 3 roods 5 perches 1 yard 7·335 feet.

PROB. XXIII. To take the plan of a field by going round it.



First, with the Plane-Table. Place the table at a corner A, and fix it when the needle points to the fleur-de-lis, and take a point A on the paper. Direct the index from the assumed point to the corner E of the field, and draw a line; then direct the index to B, and draw another line. Measure the lines in the field from A to B and from A to E, and lay these lines on the paper. Place the table at B, and, laying the index along BA on the paper, turn the table about till A is seen through the sights: the needle ought then to point to the fleur-de-lis. Direct the index to the corner C of the field, and draw a line, on which lay the length of BC. In the same manner are to be laid down the position and the lengths of the other sides CD and DE, and the last line will terminate at E on the paper, if no error has been committed.

Secondly, with the Theodolite. Place the instrument at the corner A of the field, and, having turned it till the needle points to the fleur-de-lis, take the bearing of one of the sides, as AE; then observe the angle EAB, and measure AB. Again, place the theodolite at the corner B, and observe the

angle ABC, and measure BC. And proceed in this w[ay, to] take all the angles and to measure the sides.

Add all the angles together, if they be interior; but if a[ny] of them be exterior, add the difference between it and 36[0°]; the sum should be equal to 180°, multiplied by the number [of] sides, wanting two.

If the interior angles cannot be taken, let the exterior b[e] taken by extending the direction of the sides. The sum of al[l] the exterior angles should be 360°; but if any of the corner[s] point inward, add 180° to 360° for every such angle, and the sum should be the sum of the angles.

The things measured for laying down the plan of a field will always be sufficient for finding its content, but they will not always afford the shortest method. Thus, in taking the plan of the pentagonal field ABCDE by measuring the sides and angles, if we draw diagonals AC and CE, we can find the area of the triangle ABC from the sides AB and BC and the angle B, and the triangle CDE from the sides CD and DE and the angle D; but then we have nothing given in the triangle ACE from which to find its area. We must therefore find, by trigonometry, in the triangle ABC, the angle ACB and the base AC, and in the triangle CDE, the angle DCE and the base CE; and these two angles, subtracted from BCD, will give the angle ACE, from which, with the sides AC and CE, we can find the area of the triangle ACE. And thus, by the help of trigonometry, we may find in every case sufficient data for computing the area from the things measured for taking the plan. Shorter methods are given afterwards.

1. Let AB be 750, BC 810, CD 628, DE 598 links, and the angles at B 72°, at C 136°, and at D 122°. Required the area.

The angles will be found to be ACB 50° 58′ 11″, DCE 28° 13′ 23″, and ACE 56° 48′ 26″, and AC 918·23, and CE 1072·32 links.

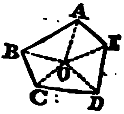
Ans. Area 8 acres 2 roods 16 perches 6 yards 4·283 feet.

2. In a six-sided field ABCDEF, let AB be 482, BC 586, CD 760, DE 812, and EF 910 links, and the angles at B 96°, at C 132°, at D 146°, and at E 106°. Required the area.

Ans. Area 15 acres 3 yards 5·122 feet.

PROB. XXIV. To survey a field from a station within it.

The station must be chosen such, that all the angles may be seen from it.



First, with the Plane-Table. Place the table at O, from which all the corners may be seen, and turn it to bring the needle to the fleur-de-

on the paper take a point O, to represent the station.
e index from O to the corner A, and draw a straight
present OA in the field. Draw, in the same manner,
present OB, OC, &c. Then measure from the station
C, &c. in the field, and lay them on their represent-
nd join their extremities.

ly, with the Theodolite. Place the instrument at the
, and, putting the needle to the fleur-de-lis, take the
OA. Next observe the angles AOB, BOC, &c.,
ded, should amount to 360°. Then measure straight
O to A, B, C, &c.

pose OA 798, OB 459, OC 434, OD 852, and OE
, and the angles at O, AOB 74°, BOC 38°, COD
E 82°, and EOA 64°. Required the area.

Ans. 11 acres 1 rood 8 perches.

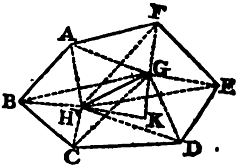
heptagonal field I found the angles at the instru-
e 67°, 43°, 84°, 56°, 27°, 51°, and 32°, and the
of the angles from the instrument to be 528, 632,
732, 830, and 816 links. Required the plan and

Ans. Area 12 acres 1 rood 6 perches 12·07 yards.

XXV. To survey a field from two stations.

tions must be such, that all the objects to be laid
the plan may be seen from them both, and that the
ich they make with the line joining the stations may
small.

with the Plane-Table.
table at one of the sta-
the needle to the fleur-
d take a point G on the
represent that station, and
sights of the index from
other station, and draw
on it lay the distance be-



stations from G to H. Direct the sights from G to
r A, and draw GA with a black-lead pencil, and
part of it place the letter A. Again direct the
n G to the corner B, and draw GB, and on it write
e same manner draw GC, GD, &c.

the table to the second station, and turn it till the
ints to the fleur-de-lis; then the index, laid on HG
er, will point to the former station. Direct now the
n H to the corner A, and draw HA, which will
line GA in the point representing that corner, at
ce A, and erase the former A. In the same manner
, meeting GB in B, and so on; then join AB, BC,

&c. In the same way the position of any other thing, as the house K, may be determined by drawing GK towards it when the table is at G, and HK towards it when the table is at H.

Secondly, with the Theodolite. Place the instrument at the first station G, and turn it till the needle points to the fleur-de-lis, and take the bearing of the station H, and measure GH. Then take the angle HGC, then CGD, DGE, &c., and lastly BGH. Remove the instrument to the second station H, and bring the needle to the fleur-de-lis; then the station G ought to bear upon the point opposite to that upon which H bore from G. If it does, then take first the angle GHF, then FHA, AHB, &c., and lastly EHG. The sum of the angles taken at each station ought to be exactly 360°.

Every thing else which is to be put in the plan must be surveyed in the same way, by taking at G the angle between GH and the line from G to it, and the same at H. All these observations must be placed in a field-book.

When the whole cannot be seen at two stations, more stations must be taken. The lines between the stations must be measured, and the angles taken as before. But care must be taken to determine the position of each of the lines joining the stations.

1. Required the plan and the area of a field from the following

FIELD-BOOK.

Angles at G.			Angles at H.			Remarks.
C	22°	0′	F	20°	0′	GH bears S. 67° 30′ W.
D	86	30	A	72	0	1038 links.
E	146	30	B	145	0	Corner of a house at K.
F	232	30	C	243	0	Angles { at G 50°.
A	313	30	D	317	0	at H 323°.
B	348	30	E	344	0	
H	360	0	G	360	0	

In this field-book, the angles at G are marked as taken with the theodolite when placed at that station. The sights, when at the beginning of the degrees, were directed to the station H, and the instrument fixed there. Then the moveable index was turned to C, and cut off 22° for the angle HGC, which, in the field-book, is marked C, the other two letters being found at the top; then it was turned to D, and cut off 86° 30′ for the angle HGD; and the difference of these two is the angle CGD. It was then turned to E, and cut off 146° 30′ for the angle HGE; and so on all the way

ground. In the same way the angles were taken at H, both for determining the corners of the field and for finding the corner of the house at K.

In calculating the areas of fields surveyed from more than one station, it is necessary to calculate, by trigonometry, the length of all the lines drawn from one of the stations to the angles; and for this purpose we have, in every triangle of which GH is a side, all the angles and this side to find the other side; after which the area is found as in the preceding problem. Here the distances from G are GA 1123·3, GB 1493·1, GC 1409·73, GD 917·43, GE 951·47, and GF 660·743 links; from which the areas of the triangles AGB, BGC, CGD, DGE, EGF, and FGA, are to be calculated.

Ans. 27 acres 5 perches 25 yards 3·47 feet.

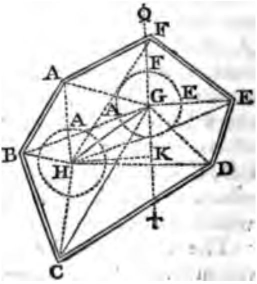
2. Required the plan and the area of a field from the following

FIELD-BOOK.

Angles at P.		Angles at R.		Remarks.
F	3°	A	6°	PR bears S. 22° 30′ E. 1827 links.
E	28	H	24	
D	49	G	64	
C	65	F	186	
B	132	E	228	
A	197	D	271	
H	247	C	319	
G	320	B	342	
R	360	P	360	

Ans. Area 100 acres 1 rood 19 perches 21 yards 1·4 foot.

Prob. XXVI. To draw the plan of the field upon paper from the field-book.



Draw a faint line up and down the paper to represent the meridian, the upper end the north, and the under end the south. Using the data given in Ex. 1, Prob. XXV., in this line take a convenient point G for the first station. On the south side of G make an angle of 67° 30′ towards the left hand, which will give the position of GH; and take 1038 from any convenient scale, and lay that extent from G to H,

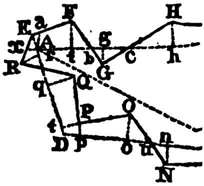
to get the station H. The best protractor for laying the angles is a circular one, divided into 360°. Pla centre at G, and the beginning of the degrees on GH. a mark at 22°, and at it write a faint C; make anothe at 86° 30′, and there write a faint D, and so on all th round; and draw faint lines from G to the marks. place the centre of the protractor at H, and the begin the degrees on GH; and at 30° make a mark, and wr at 72° make a mark, and write A, and so on; and dra from H through the marks. The lines from G a through the points where the same letter is written, drawn out till they meet, and their intersection is angle to which that letter belongs. Thus GA and H meet in the angle A, GB and HB will meet in the angl After this join AB, BC, &c. for the boundaries of the

If the protractor be a semicircle, then, after layin the angles less than 180°, the protractor must be laid other side of GH, and 180° taken from each of the re angles before they are laid down.

Prob. XXVII. To survey fields with boundaries.

The boundaries of fields are seldom straight lin therefore surveyors generally erect poles near the co the ground to be surveyed, and conceive these poles jo straight lines. This constitutes the body of the fiel the parts between these lines and the boundaries are co as offsets, and their areas found separately.

The points, therefore, which, in the preceding pr were called angles or corners, are to be considered onl places of these poles, and the fields surveyed as contai the lines joining them; and to complete the survey, th ation and distance of the boundaries from these lines found.



1. Let EIMP be a field to be surveyed. Poles are erected at A, B, C, D, near corners of the field, and the space ABCD is surveyed as before. The rest of the field is obtained by taking offsets from the lines AB, BC, CD, DA, and adding the spaces which are without these lines, and taking away th within them.

The field-book for such a survey must consist columns: the middle one contains the distance measur

FIELD-BOOK.

Left offsets.	Main lines.	Right offsets.
	AC, S. 60° 25′ E. 1896.	
	844	Including offset to cor.
86	746	Close to A.
152	688	
	594	
	462	200
D	64	90
	1410	D Γ
	1362	92
	924	196
	744	
146	600	
C 48	0	
108		C Γ
104	912	
264	508	
84	152	
B 70	0	
128		B Γ
94	1672	
172	1166	
	752	
	530	108
	442	
200	284	
A 106	0	
To left.		To right.

ıes AB, BC, he other two ɪffsets, accord- ·e on the right he main line. pose it is best the bottom of r, and to write at the offsets ; side of the ıay be placed hand column, ts on the left left-hand co- in measuring B, the offset neasures 106 the left hand he beginning herefore write ddle column, , and opposite left-hand co- 106. Then ong AB, the e found, upon perpendicular : this is 284 ı, and *f*F is ıerefore write iddle column, osite to it in column. A- links from A, crosses the boundary-line FG; therefore write ıiddle column, and in the adjacent columns draw in the direction of the straight line FG nearly, position of it is not required at this stage of the 530 the perpendicular from G meets AB, and ɔlace therefore 530 in the middle column, and to it in the right-hand column.

this way to B, where, besides the offset, BI is ɪ placed in the left-hand column, with the mark hat it is not perpendicular. At the same place ıand column is placed the mark Γ, to show that ɪyor turns to the right hand. This finishes the *he line AB*, and a line is drawn across the book

to separate it from the next line. Proceed in the same w
from B to C, from C to D, and from D to A.

The position of any one of the lines, as AC, being fou
with the compass, it will determine the position of the wh
But in using the compass, the variation should be allowe
and great care ought to be taken lest the needle be attract
by some metallic substance in its neighbourhood.

Ans. Area 14 acres 2 roods 19 perches $22\frac{1}{3}$ yar

(2.) FIELD-BOOK.

Left off-sets.	Main lines.	Right off-sets.
Diagonal AC, N. 28° W. 760 links.		
0	660	
30	450	
D 0	400	
0	490	D ┌
10	400	
40	300	
55	200	
C 20	50	
	635	0 C ┌
	500	25
	400	30
	300	
50	200	
B 40	100	
0	395	B ┌
20	350	
35	300	
45	250	
50	200	
30	100	
A 15	50	

Ans. 3 ac. 28 per. 7·038 yds.

(3.) FIELD-BOOK

Left off-sets.	Main lines.	Right o sets.
Diagonal AC, S. 56° E 1560 links.		
	1350	
0	1200	
40	900	
20	750	
60	550	
85	400	
70	350	
D 35	200	
0	800	D
34	700	
	500	
	350	80
C	200	60
B	1100	C
0	912	B
40	800	
	750	
	680	50
	600	
90	450	
A 50	340	

Ans. 10 ac. 3 ro. 10
17 yds. 5·558 feet

Lay down the plans of the following properties fron
field-book for the three examples, and calculate their con

Fig. 1.

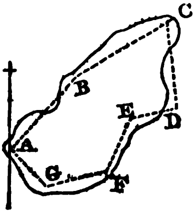
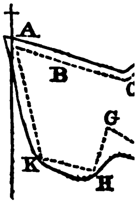


Fig. 2.



(4.) (Fig. 1.)

Diagonals.
BD 1100
BE 720
BF 1080
AF 1000
B bears N. $37\frac{1}{4}°$ E.

	510	A ┌
	360	
	0	
	612	G ┌
	320	
	0	
	600	F ┌
	256	
	0	
	480	E ┐
	220	
	114	
	0	30
┌	920	36
	826	78
	560	340
	356	90
	281	
	180	
	0	
	900	C ┌
	728	
	560	
	256	
	0	
	1040	B ┌
	980	
	826	
	673	56
	522	
	443	
	156	
	0	A

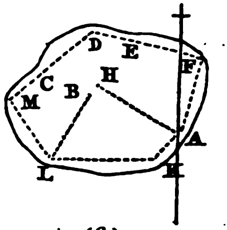
15 ac. 5 per. 15 yds.
3·394 feet.

(5). (Fig. 2.)

Diagonals.
CE 620
CF 1000
CG 610
GB 850
GK 710
BK 940
AK bears S. 11° E.

20	1150	A ┌
25	680	
35	420	
50	0	
60	580	K ┌
90	500	
150	300	
100	0	
89	470	H ┌
130	260	
200	0	
400	800	G ┐
380	630	
220	480	
36	230	
	153	
	110	25
	0	40
F ┌	760	50
	640	78
	520	115
	380	85
	200	40
	86	
30	0	
30	420	E ┌
35	320	
30	100	
20	0	
25	500	D ┌
89	360	
72	150	
30	0	
40	730	C ┐
150	540	
110	210	
30	0	
20	450	B ┐
70	250	
30	0	A

Ans. 18 ac. 1 ro. 23 per.
25 yds.



(6.)

	2180	A	S. 59° E.
15	626	15	
	426	H	
20	0	10	
20	1610	10 B Γ	N. 29° E.
20	1590		
	0	L	
To houses.			
A Γ	2050	15	S. 13° W.
	1969		
180	1000		
9	0		
51	1380	F Γ	S. 77° E.
120	600		
20	0		
20	750	E Γ	S. 85° E.
24	500		
10	0		
10	1400	D Γ	N. 51° E
500	1000		
400	700		
300	400		
	25		
	0	20	
15	655	C Γ	
10	0		N. 45° E.
10	1450	M Γ	N. 31° W.
350	600		
20	0		
20	2280	L Γ	N. 85° W.
220	1400		
10	0		
10	640	K Γ	N. 36° W.
100	400		
20	0	A	

Ans. 89·26 acres.

ROB. XXVIII. To take an extensive survey.

hoose for stations the most eminent places, from which the cipal parts of the survey may be seen. Particularly choose eminences as lie near the boundaries. Take the angles ch these stations make with one another with great accu-, and measure carefully in a straight line the distances station to station, marking the places where the lines ditches, roads, rivulets, &c., and take offsets to near cts, leaving in the ground a mark at every place where marked the distance in the field-book, distinguishing these ks by letters or figures, that they may not be mistaken for another. In this way you will obtain the situation of the cipal parts. Then take other stations within these, and sure the distances as before. And thus divide and sub-de the survey, till you come to single fields, which may be sured by some of the preceding methods.

'he longer the distance is between the stations, if accu-ly measured, the more correct will the work be; but this not be ascertained by a single measurement, without using ous methods of determining it. At the same time, an r in these primary distances affects the whole survey; and refore every care ought to be taken to prevent it.

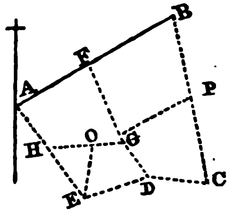
After the principal parts of the survey are laid down accu-ely, so as to have the whole divided into small compart-nts, these may be filled up by the plane-table, one by one.

In laying down the plan, proceed in the same way, first ing down the principal distances and the boundaries, and n the interior parts as they are surveyed; and in filling up particular departments, care must be taken to lay down boundaries of parishes, estates, farms, &c. and to point out particular situations of towns, villages, churches, gentle-n's seats, towers, farm-steads, also rivers, lakes, ponds, ods, plantations, rocks, precipices, and all the eminences, nes, pits, quarries, and in general every thing which can tribute to give a proper understanding of the nature of the vey. All these must be neatly sketched and properly oured, and the names of the places are to be printed in them.

1. I took two stations near a road, of which B lay from A, 61° E. 1850 links; and from A took the bearings of the inences C, S. 70° E., D, S. 62° E., and E, S. 36° E., and B took their bearings C, S. 14° E., D, S. 6½° W., and E, 26° W. Required their distances from the stations, and ir bearings and distances from one another.

Ans. BC 1684·14, AE 1201·788, CD 596·64, and D 753·41 ks.

Having drawn the plan of the observations in Ex
it is required to lay down on it, and to calculate the pr
contained in the field-book of the following examples.



(2.)

Diagonal. FH 935		
35	560	A
100	320	
88	180	
20	0	
20	695	H ⌈
60	513	
	313	
O	300	4
	0	5
	870	G ⌈
105	450	
50	0	
4	900	F ⌈
98	734	
150	540	
122	330	
40	0	A
	At the road.	

Ans. 7·28677 acres.

(3.)

Diagonal. PF 1065		
G	945	5
	878	80
	805	P
44	366	
10	0	
10	950	B
28	825	
90	740	
60	580	
30	430	
30	400	
78	260	
20	0	F
	At the road.	

Ans. 7·30361 acr

(4.)

Diagonal. PD 945		
	540	G
	360	58
	260	80
	0	20
0	597	D Γ
8	350	
	0	
	879	C Γ
3	621	
0	421	
	0	P

(5.)

Diagonal. EG 670		
4	564	O Γ
70	372	
130	248	
65	100	
12	0	
12	753	E Γ
90	613	
160	518	
170	416	
150	298	
40	0	D

ns. 6·50322 acres. Ans. 4·07145 acres.

e distances not mentioned in these two examples are to
en from the preceding ones.

OB. XXIX. To find the contents of a survey.

e areas of single fields, bounded by straight lines, may be
from the lines measured in the field, by the first twelve
ms of MENSURATION OF SUPERFICIES.

CALCULATE OFFSETS. The most accurate method is
npute them separately, as triangles and trapeziods, by
IV. and VII. of MENSURATION OF SUPERFICIES.

THOD 2. If the distances between the perpendiculars be
equal. To half the sum of the perpendiculars at the
nities of the base, add all the rest, and multiply the sum
base, and divide the product by the number of divisions
base made by these perpendiculars.

MMON METHOD. Divide the sum of the perpendiculars
number of them for a mean perpendicular, by which
ly the base.

The fourth Example in Prob. XXII. wrought by the first Method.

50×110	$= 5500$	for the	triangle	AKa
$170 \times (110 + 135)$	$= 41650$	. . .	trapeziod	KabH
$75 \times (135 + 85)$	$= 16500$	. .	trapeziod	HbdG
$85 \times (85 + 275)$	$= 30600$	. .	trapeziod	GdfF
$275 \times (275 + 185)$	$= 126500$	. . .	trapeziod	FfiE
70×185	$= 12950$	. .	triangle	EiD
145×75	$= 10875$	. . .	triangle	ABc
$330 \times (75 + 150)$	$= 74250$	. .	trapeziod	BceC
250×150	$= 37500$	. . .	triangle	CeD

$2)\overline{356325}$

Ans. $\overline{178162{\cdot}5}$ the whole area.

By the second Method.

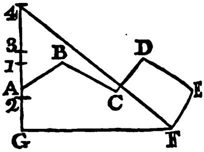
Ans. $725 \times (\frac{1}{6} \times (110 + 135 + 85 + 275 + 185) + (150 + 75) \times \frac{1}{3}) = 149833\frac{1}{3}$ area.

By the third Method.

Ans. $725 \times (\frac{1}{5} \times (110 + 135 + 85 + 275 + 185) + (150 + 75) \times \frac{1}{2}) = 196112{\cdot}5$ area.

But surveyors generally endeavour first to obtain a correct plan of the land, and then they measure, on the plan, such lines as will enable them to calculate its contents with the greatest expedition; and for this purpose they reduce the crooked boundaries to straight lines. Sometimes this is done by stretching a hair through the crooked part, so that the small parts cut off by the hair may be equal to the parts taken in, as nearly as the eye can judge; and this can be done very nicely by an experienced surveyor.

Others reduce the crooked parts to a triangle, by Prob. XXXIV. of Practical Geometry, which can be done by the parallel ruler without drawing lines. Thus, suppose ABCDEFG to be the space which is to be reduced to a triangle. Lay the parallel ruler from A to C, and move it till it pass through B, and mark the point 1 in which it cuts AG. Lay the ruler through 1 and D, and move it till it pass through C, and mark 2 where its cuts AG.

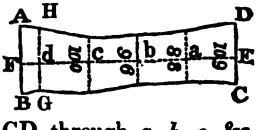


Again lay the ruler from 2 through E, and move it till it pass through D, and mark 3 where it cuts AG, and so on; then join 4 and F, and the triangle F4G is equal to the given space. For B1 is parallel to AC; therefore if C1 were drawn, the triangle AC1 = ACB. Now, when the ruler passes through A and C, it takes in the triangle ACB; and when

; moved to B1, it cuts off the triangle AC1. In like
ıner the triangle 1D2, which is taken in, is equal to the
ngle 1DC cut off; and so of the rest.

ınother method of calculation much practised by surveyors
ıe following, which, though it depend upon judgment, will
ound to come very near the truth, and is very expeditious.

ıet ABCD be the plan of a
'ey, and DC a straight bound-
. Draw EF perpendicular
DC, and on it lay a chain,
ı E to *a*, from *a* to *b*, from



c, &c.; and draw parallels to CD through *a*, *b*, *c*, &c.,
they will divide the plan into spaces, each a chain in
ıdth. Measure in a line parallel to DC, half-way between
nd *a*. This is supposed to give the mean length of the
space, and therefore is to be measured where the length
mean, as nearly as the eye can judge. It is here supposed
e 109 links, and is written so in the first space. In the
e manner the mean lengths are taken in all the other divi-
s. After this these lengths are to be added together, and
ire only three places to be cut off to give the area in acres.

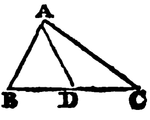
small space ABGH remaining beyond the last parallel,
h is only 39 links in breadth, may be found by multi-
ng 39 by its mean length, judged of as before. Or offsets
ı GH may be taken from A and B, and thus a mean
dth may be obtained, to be multiplied by GH, or the
n length. Suppose the offsets at A and B to be 44 and
and suppose the mean length to be 96 links; then
< 39 = ·03744 of an acre. Or the mean offset is 37·5,
ch, multiplied by GH, suppose 100, gives ·03750 of an
e for the content of the part ABGH; and this, added to
3, the sum of the mean lengths of the other pieces, gives
05 of an acre, or 1 rood 28·88 perches, for the whole area.

f the boundary be a curve line, and the distances between
perpendiculars equal, the area may be calculated by Note 2,
b. XXX. of MENSURATION OF SUPERFICIES.

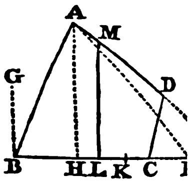
OF DIVIDING LAND.

'ROB. XXVI. To divide a triangular field ABC in
proportion, as that of 9 to 7, by a straight line
wn from the angle A, the opposite side BC being
links.

ns. 16 : 7 :: 950 : 415$\frac{5}{8}$ to be laid from
D; then AD is the dividing line.



Divide the triangle ABC, of which the
are AB 386, BC 428, and AC 533

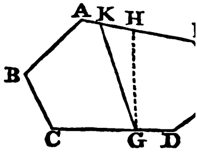


Consider which of the sides are to be cut by the dividing line, as AD and BC. Produce these lines till they meet in F. Reduce the field to a triangle ABE, and divide BE in the given ratio. Draw AH in the position of the dividing line, and make FL equal to the square root of the product of FK and FH, and LM parallel to AH, and it will be the dividing line.

2. Divide the quadrilateral figure ABCD, of which . 356, BC 528, CD 216, and AD 418 links, and the ABC 78°, in the ratio of 3 to 4, by a line perpendicu BC. Ans. BL 205, or CL 323

These methods of dividing land, though accurate, a general as short as any, are seldom used by surveyors. generally draw a line as near as they can conjecture t position of the dividing line required, and find the area c part cut off by it, which discovers how much they hav off, too little or too much; and they alter the line as i following problem.

Prob. XXXII. From a given field ABCD suppose of 20 acres, to cut off 8 acres towards B, straight line drawn from the point G in the line 436 links from C.



Draw GH as near to the position of the line required as you can conjecture, and calculate the area of the space ABCGH, which suppose to be 9·0496 acres: this is 1·0496 acres too much, which, divided by $\frac{1}{2}$GH, suppose 364 links, gives 288 links. Draw a parallel to GH at this d: from it, and let it meet AF in K: then GK is the di line required.

When a quantity of land, such as a common, is to be d among several proprietors in certain proportions, the qu to be assigned to each will be as the value of his clai vided by the quality or value of the ground allotted t This may be done by adding into two sums the conten the values: then, by distributive proportion or fello compute the value of each person's share; and from the lity of the ground where his share is to be determined what quantity will amount to the value of his share, a it off by the last problem.

uppose it were required to divide 780 acres among three prietors, whose estates are £1000, £3000, and £4000 ear, and the values of the land in which their shares are to are 5s., 8s., and 10s. the acre respectively.

'he claims being as 1, 3, and 4, and the qualities as 5, 8, 10, the quantities assigned to them must be as $\frac{1}{5}$, $\frac{3}{8}$, and $\frac{4}{10}$, s 8, 15, and 16, and their shares 160, 300, and 320 acres.

ROB. XXXIII. To transfer, and to enlarge or iinish, a plan.

After the first plan is completed, it will be necessary to v out a fair one upon vellum or paper.

There are various ways of doing this.

First. If the fields be generally bounded by straight lines, the plan upon the clean paper, keeping it firm by weights, prick through all the corners of the plan, and then con- the points on the clean paper.

Secondly. Lay a piece of paper covered with black-lead between the papers, with the powdered side towards the n paper, and with a blunt needle trace all the lines on the gh plan with such pressure that the impression may reach clean paper; after which they are to be traced with ink n the clean paper.

Thirdly. Divide the rough plan into small squares, and de the paper to which it is to be transferred into as many ares; then copy the parts of the plan found in each square the corresponding square of the other plan. In this nner the plan may be enlarged or diminished in any pro- tion, by making the squares in that proportion.

Fourthly. There are several instruments useful for trans- ing, enlarging, and diminishing plans, as the proportional passes, the pentagraph, and the copying-glass.

A plan may be enlarged or diminished in any proportion the first paper, by Prob. XXXVII. of PRACTICAL GEO- TRY, and afterwards transferred to the clean paper by any he preceding methods.

After the plan is copied upon the clean paper, write such es, remarks, or explanations as are reckoned to be neces- , and make a fleur-de-lis to point out the direction, and convenient corner lay down a scale for measuring the s of the plan. The title of the plan must be placed in nspicuous part, and properly ornamented. After which, y part must be coloured or illuminated in the way that ears most natural. Rivers, woods, hills, hedges, houses, ls, &c. must all be distinguished by proper representations. these things require to be learned by practice.

GAUGING.

GAUGING is the method of taking the dimensions o vessel, and of finding the quantity of liquor in it.

In Mensuration the dimensions are taken on the ou but in Gauging they are taken on the inside of the vesse

The dimensions of vessels are taken in inches, and t fore the content may be found in inches by the Rules fo MENSURATION OF SOLIDS; after which they may be re to gallons, bushels, or pounds, by the following

TABLE.

Cubic inches.	
277·274	1 imperial gallon for all goods.
2218·192	1 imperial bushel.
2273·461	1 imperial bushel ground malt.
25·67	1 pound green soft soap.
25·56	1 pound white soft soap.
27·14	1 pound cold hard soap.
30·28	1 pound tallow, gross.
34·8	1 pound starch.
8·46	1 pound green glass.
9·178	1 pound plate glass.
10·516	1 pound broad glass.

OLD MEASURES.

231	1 gallon of wine, spirits, oil, &c.
282	1 gallon of beer or ale.
268·8	1 corn gallon.
2150·42	1 malt bushel.
2204	1 bushel ground malt.
104·2034	1 Scotch pint.
2214·322	1 firlot wheat, pease, rye, and sal
3230·305	1 firlot barley, oats, and malt.
271·25	1 Irish gallon.
8680	1 Irish barrel.

Gauging is, for ease, generally performed by the sl rule.

DESCRIPTION OF THE SLIDING-RULE.

his rule is 1 foot long, 1·1 inch broad, and ·8 inch thick, each of its four sides is furnished with a slider.

pon the first side are four lines, all constructed in the way, that is, each is divided into 1000 parts, and the bers are placed at their logarithms. The two on the r are marked B. or Num. The upper line on the rule is ked A., and the under one M. D. or Malt Depth. This is inverted, and the point 2218·192 is placed at the right so that 10 on this line is opposite to 2218·192 or IM. B. he line A.

pon the opposite side of the rule, the lines on the slider he same with those on the first side. The line on the is constructed in the same way, only the distance between d 10 is twice as long. One-half of this line, or from 1 2, is placed above the slider, and the other half below it. e rules have a line on which the distance between 1 and one-third of the distance on this line. On the inside of lider are the gauge-points for imperial gallons, for impe- malt bushels, and also the multipliers and divisors for re and round vessels.

n the other sides or edges of the rule, the sliders contain the same with those of the other sliders; and on the rule ines for ullaging, on the one edge for ullaging a lying and on the other for a standing cask. These lines are ructed experimentally thus:—Take a cask containing gallons, and fill it with water. Draw off one gallon, and ure the depth of the remaining water; then set the h, or the bung-diameter, according as the cask is stand- r lying, on the slider, opposite to 100 on the rule, and site to the wet inches on the slider mark 99 on the rule. v off another gallon, measure the wet inches, and opposite em on the slider mark 98 on the rule; and proceed in same manner till the line is all marked. The inside of slider, on the edge marked C, contains a line of inches lines for reducing the first and second varieties of casks to ders.

here are several brasses or notches marked on the lines. s, on the first side, a brass with IM. B. is marked at 3·192 for imperial bushels, and another with IM. G. for erial gallons at 277·274. On the second side are marked he rule the gauge-points, IM. G. for imperial gallons at '89, M. S. or malt bushels in square vessels at 47·097, and R. or malt bushels in round vessels at 53·144.

Prob. I. To multiply by the sliding-rule.

Turn up the first side, and set 1 on the slider opposite to the multiplier on the line A; then against the multiplicand on the slider is the product on A.

1. Multiply 15 by 8. Set 1 on B to 8 on A; then opposite to 15 on B will be 120 on A, the product.

Note. The 1 at the left end of A may be read 1, or 10, or 100; and the rest of the numbers must be read according; the 2 either 2, or 20, or 200, &c. Also, in reading the multiplicand on the slider, the 1 may be read 10 or 100; but then the product must be increased 10 or 100 times.

2. Multiply 250 by 56. Set 1 on B to 56 on A; then against 250 on B is 14000 on A.

3. Multiply 7·23 by 8·5. . . . Ans. 61·4[illegible]
4. . . 82·5 by ·73. . . . . 60·22
5. . . . ·94 by 7·4. . . . 6·9[illegible]

Prob. II. To divide by the sliding-rule.

Place the divisor on the slider B opposite to the dividend on A; then against 1 on B is the quotient on A.

1. Divide 480 by 15. Set 15 on B to 480 on A; then against 1 on B is the quotient 32 on A.

2. Divide 8142 by 59. . . . Ans. 138.
3. . . 8·75 by 3·25. . . . . . 2·69.
4. . . 6·08 by 7·42. . . . ·819
5. . . 19·7 by 3·5. . . . 5·63.

Prob. III. To work a proportion by the sliding-rule.

Place the first term on the slider B opposite to the second or third on A; then against the other term on B is the answer on A.

1. If 40 yards of cloth cost £24, what will 15 cost?

Ans. Set 40 on B to 15 on A; then against 24 on B is £9 on A, the answer.

2. How many yards of cloth at 18s. may be given for 60 lb. of tea at 7s.? Ans. $23\frac{1}{3}$ yards

3. If 16 men do a piece of work in 48 days, in what time will 24 men do it? Ans. 32 days

4. What number of men must be employed to perform in 84 days a piece of work which 108 men perform in 133 days? Ans. 171 men

5. If £15·6 pay 16 labourers for 18 days, how many, at the same rate, will £35·1 pay for 24 days? Ans. 27 labourers

. If 36 yards of cloth, 7 quarters wide, cost £25·2, what 120 yards of the same quality, 5 quarters wide, cost?

Ans. £60.

Prob. IV. To extract the square root by the sliding-e.

Take the second side of the rule. Place 1 on the slider osite to 1 on the rule, then find the given number on the er, and if it consist of 1, 3, 5, 7, &c. figures, the root is osite to it on the line above; but if it consist of 2, 4, &c. figures, the root is opposite on the line below it, on the .

. Required the square root of 81. Set 1 on C to 1 on D; n opposite to 81 on C, is 9 on the line below on D.

. Required the square root of 625. Set 1 on C to 1 on D; n against 625 on C, is 25 on D on the line above.

. Required the square root of 1681. . Ans. 41.

. . . . of 24649. . . 157.

. . . . of 5·0625. . . 2·25.

. . . . of 30·25. . . 5·5.

Prob. V. To find a mean proportional between two mbers.

Set the lesser on C to the lesser on D; then against the ater on C is the mean proportional on D.

1. Required a mean proportional between 18 and 72.

Set 18 on C to 18 on D; then against 72 on C is 36 on D, ich is the mean required.

2. Required a mean proportional between 2448 and 17.

Ans. 204.

3. Required a mean proportional between 128 and 1152.

Ans. 384.

4. Required a mean proportional between 30·25 and 272·25.

Ans. 90·75.

5. Required a mean proportional between 1248 and 78.

Ans. 312.

6. Required a mean proportional between 205·5 and 137.

Ans. 167·79.

Prob. VI. To find a number, which shall have to a en one the same ratio which the squares of two given mbers have to one another.

Set the first term of the ratio on D to the given number on then opposite to the other term of the ratio on D stands answer on C.

1. Required the number which shall be to 36, as the square of 4 to that of 3.

Set 3 on D to 36 on C; then against 4 on D will be 64 on C, the answer.

2. What number is to 120, as the square of 3 to that of 2? Ans. 270.

3. Increase the number 240 in the ratio of the square of 4 to that of 5. Ans. 375.

4. Diminish the number 392 in the ratio of the square of 7 to that of 6. Ans. [illegible]

5. Find the number to which 196 shall have the same ratio with the square of 7 to that of 9. Ans. [illegible]

Prob. VII. To find a number which shall be to a given one as the square roots of two given numbers.

Set the first term on C to the given number on D; then against the other term on C stands the answer on D.

1. To what number will 3 have the same ratio with the square root of 108 to that of 48?

Set 3 on D to 108 on C; then against 48 on C is 2 on D, the answer.

2. To what number will 2 be as the square root of 120 to that of 270? Ans. [illegible]

3. Required the number to which 256 shall be as the square root of 16 to that of 9. Ans. [illegible]

4. Increase the number 433 in the ratio of the square root of 3 to that of 5. Ans. [illegible]

5. Diminish the number 1414 in the ratio of the square root of 8 to that of 7. Ans. [illegible]

Prob. VIII. Of multipliers, divisors, and gauge-points.

Instead of first finding the content of a vessel in inches, and afterwards reducing it to the measure of capacity required, which must often be done both by multiplying and dividing by known numbers, gaugers find the content in the measure required by means of a single multiplier or divisor.

These multipliers are got by dividing the multiplier used in finding the content by the divisor, which reduces the content to gallons, &c. Thus, to find the multiplier which, in circular vessels, will give the content in imperial gallons, divide ·785398 by 277·274.

To find the divisor which will answer the same purpose divide 277·274 by ·785398.

Gauge-points are numbers made use of in working by the

le. The operation is made similar to that in ; and for that purpose the square root of the divisor or the first term, and is called the Gauge-point.

TABLE I.

IPLIERS, DIVISORS, AND GAUGE-POINTS, FOR CYLINDRICAL VESSELS.

asures.	Multipliers.	Divisors.	Gauge-Points.
ies, . .	·7853982	1·273239	1·12838
l gallons, .	·0028326	353·0362	18·7893
l bushels, .	·0003541	2824·2903	53·1441
oft soap, lbs.	·0305959	32·6841	5·7170
oft soap, do.	·0307276	32·5440	5·7047
rd soap, do.	·0289388	34·5557	5·8784
gross, do.	·0259379	38·5537	6·2092
do. . .	·0225689	44·3087	6·6565
lass, do.	·0928367	10·7716	3·2820
ass, do. .	·0855740	11·6858	3·4184
lass, do. .	·0746860	13·3894	3·6592
IEASURES.			
allons, . .	·0034000	294·1183	17·1499
ons, . .	·0027851	359·0535	18·9487
llons, . .	·0029219	342·2468	18·4999
ishels, . .	·0003652	2738·0000	52·3259
pints, . .	·0075372	132·6759	11·5185
irlots, . .	·0003547	2819·3623	53·0977
firlots, . .	·0002431	4112·9526	64·1323
llons, . .	·0028955	345·3662	18·5840
rrels, . .	·0000905	11051·7176	105·1296

table, the first multiplier is that for finding the area e, and its reciprocal is the first divisor. The other rs are got by dividing the first by the number of a gallon, bushel, &c.; and the other divisors by ing the first divisor by the number of inches in a tc. The gauge-points are the square roots of the

put instead of ·7853982 at the top, tables may be the same way for square vessels. Thus, 1 divided by res ·036846, the multiplier for hard soap.

TABLE II.

MULTIPLIERS, DIVISORS, AND GAUGE-POINTS, F PRISMATIC VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-I
SQUARE.			
Imperial gallons,	·0036065	277·274	16·65
Imperial bushels,	·0004508	2218·190	47·09
Hard soap, pounds,	·0368460	27·140	5·20
Tallow, do. . .	·0330251	30·280	5·50
Starch, do. . .	·0287356	34·800	5·89
Green glass, do. .	·1182033	8·460	2·90
PENTAGONAL.			
Imperial gallons, .	·0062050	161·1610	12·695
Imperial bushels, .	·0007756	1289·2884	35·906
Hard soap, pounds,	·0633927	15·7747	3·971
Tallow, do. . .	·0568190	17·5998	4·195
Starch, do. . .	·0494390	20·2269	4·497
Green glass, do. .	·2033661	4·9172	2·217
HEXAGONAL.			
Imperial gallons, .	·0093700	106·7228	10·33
Imperial bushels, .	·0011726	853·7824	29·21
Hard soap, pounds,	·0957287	10·4462	3·23
Tallow, do. . .	·0858018	11·6548	3·41
Starch, do. . .	·0746574	13·3945	3·65
Green glass, do. .	·3071012	3·2563	1·80
HEPTAGONAL.			
Imperial gallons, .	·0131059	76·3918	8·73
Imperial bushels, .	·0016382	610·4142	24·70
Hard soap, pounds,	·1338951	7·4685	2·73
Tallow, do. . .	·1200103	8·3326	2·88
Starch, do. . .	·1044228	9·5765	3·09
Green glass, do. .	·4295405	2·3281	1·52
OCTAGONAL.			
Imperial gallons, .	·0174139	57·4253	7·57
Imperial bushels, .	·0021767	459·4027	21·43
Hard soap, pounds,	·1779081	5·6209	2·37
Tallow, do. . .	·1594592	6·2712	2·50
Starch, do. . .	·1387479	7·2073	2·68
Green glass, do. .	·5707359	1·7521	1·32

To find the multiplier, divisor, and gauge-point, for im gallons in vessels of the form of a regular heptagon.

le the tabular multiplier 3·6339124 by 277·274: the t ·0131059 will be the multiplier. Divide 277·274 by 124: the quotient 76·3918 will be the divisor, and its root 8·7351 will be the gauge-point.

he same manner the multipliers, divisors, and gauge-ire found for any regular polygon.

TABLE III.

LTIPLIERS, DIVISORS, AND GAUGE-POINTS, FOR CONICAL VESSELS.

Measures.	Multipliers.	Divisors.	Gauge-Points.
ıches, . .	·2617994	3·819717	1·59441
ial gallons, .	·0009442	1059·1086	32·5441
ial bushels, .	·0001180	8472·8708	92·0490
oap, pounds,	·0101986	98·0522	9·9021
soft soap, do.	·0102425	97·6320	9·8810
soap, do. .	·0096463	103·6671	10·1817
r, do. . .	·0086460	115·6611	10·7547
ı, do. . .	·0075230	132·9261	11·5295
glass, do. .	·0309456	32·3148	5·6846
glass, do. .	·0285247	35·0575	5·9208
glass, do. .	·0248953	40·1683	6·3379
MEASURES.			
gallons, . .	·0011333	882·3549	29·7045
allons, . .	·0009284	1077·1605	32·8201
bushels, . .	·0001217	8214·0000	90·6306
ı pints, . .	·0025124	398·0277	19·9506
t firlots, . .	·0001182	8458·0870	91·9680
y firlots, .	·0000810	12338·8578	111·0812

ıyramidal, conical, &c. vessels, where, in finding the ;, we multiply by one-third of the length, the multi-ıould be one-third of that in the table, the divisor must e times as large as that in the table, and the gauge-ıust be the square root of three times the tabular divi-ıd, in this case, use the whole length, instead of one-f it. The same remarks are applicable to rules in which ltiply by any other part of the length.

ıB. IX. To gauge areas one inch deep.

Vhen one side is given, set the gauge-point on D to ; and against the given side on D is the answer on C.

T 2

1. Suppose the side of a square to be 77 inches. Requi its content, at 1 inch deep, in old wine and ale gallons.

Here the multipliers are ·003546 and ·004329, the divi are 282 and 231, and the gauge-points 16·7929 for ale, 15·1987 for wine gallons.

77	231)5929	282)5929
77	25·667 wine gal.	21·025 ale g
5929 content in inches.		5929
·003546		·004329
21·024 ale gallons.		25·667 wine gallons.

By the Gauge-Points.

Set the gauge-point for ale, 16·7929 on D, to 1 on C; th against 77 on D will be found 21 ale gallons on C.

Set the gauge-point for wine, 15·1987 on D, to 1 on and against 77 on D is 25·7 on C, the wine gallons.

2. Required the content, in imperial gallons, of a squ vessel at 1 inch deep, the side 98 inches. Ans. 34·6368 g

3. Required the content of a regular pentagon 1 inch de in hard soap and starch, the side 53 inches.

Ans. 178·07 lbs. hard soap, 138·874 lbs. sta

4. Required the content of a regular octagon 1 inch d in tallow, the side 83 inches. Ans. 1098·5144

II. When two dimensions are given, it is necessary working by the gauge-points, to find a mean proportional tween the two factors, and to work with it by the prece rule.

1. Required the content, at 1 inch deep, of a rectang vessel, of which the length is 100½ inches, and its breadtl inches, in imperial bushels and pounds of hard soap.

100·5	2010
20	·0368
2010 inches.	74·06 lbs. hard soap.
·0004508	
·906108 of an imperial bushel.	

By the Sliding-Rule.

Set 100½ on B to 1 on MD; then against 20 on A stand ·906 of an imperial malt bushel on B.

Set 27·14 on A to 20 on B; then against 100·5 on A stand 74·1 lbs. on B.

By the Gauge-Points.

First find a mean proportional between the breadth and gth.

Set 20 on C to 20 on D; then against 100½ on C will be ·83 on D, the mean proportional.

Set 47·098 on D to 1 on C; and opposite to 44·83 on D ll be ·906 of an imperial malt bushel on C.

Set 5·21 on D to 1 on C; and against 44·83 on D is 74·1 on C.

2. Required the content, at 1 inch deep, of a parallelogram, tallow and hard soap, the sides being 96 and 48, and the rpendicular upon the former 36 inches.

Ans. 114·1348 lbs. tallow, 127·34 lbs. hard soap.

3. Required the content, at 1 inch deep, in starch and een glass, of a triangular vessel, the base being 118 inches, d the perpendicular upon it 72 inches, and one of the angles 9°. Ans. 122·0688 lbs. starch, 502·1276 lbs. green glass.

4. Required the content, in imperial gallons and bushels, at inch deep, of a vessel in the form of a trapeziod, the parallel les 68 and 142, and their perpendicular distance 76 inches.

Ans. 28·77987 gallons, or 3·5975 bushels.

5. Required the content, in pounds of starch, of a trapezium, of which the diagonal is 78, and the perpendiculars on it 23 and 15½ inches. Ans. 43·1465 lbs.

Here the multiplier is ·0287356, the divisor 34·8, and the gage-point 5·899. $23 + 15{\cdot}5 = 38{\cdot}5$.

Set 34·8 on A to 39 = ½ of 78 on B; and against 38·5 on will be 43·146 lbs. on B.

6. Required the content, in old wine and ale gallons, of a gular octagon, of which the side is 42 inches.

Here the multipliers are ·0209023 and ·0171221, the visors are 47·8417 and 58·4041, and the gauge-points 9168 for wine, and 7·6423 for ale gallons.

42 × 42 = 1764		47·84)1764(36·87 wine gal.
1764	1764	58·404)1764(30·203 ale gal.
·017122	·020902	
)·203208 ale gal.	36·871128 wine gal.	

By the Gauge-Points.

Set 7·64 on D to 1 on C; then against 42 on D is 30·2 ale llons on C.

Set 6·92 on D to 1 on C; and at 42 on D is 36·9 wine llons on C.

7. Required the content, in imperial bushels, of a regular hexagon, of which the side is 138 inches.

Ans. 22·331 bushels.

III. If the inches in any of the gauge-points be laid on a rule, and this distance be divided into 100 equal parts, the dimensions may be taken with that rule, and then the content may be found without using the multipliers or divisors. Thus, if 7·64 inches be divided into 100 equal parts, the side of the octagon in Ex. 6, measured by this rule, would be 5·496 ale gallons; and this, multiplied by itself, would give 30·206 ale gallons for the content.

1. Required the content, in imperial gallons, of a circle, of which the diameter is 40 inches.

Ans. $1600 \times \cdot0028326 = 4\cdot52316$ imperial gallons.

By the Gauge-Points.

Set 18·8 on D to 1 on C; then against 40 on D is 4·53 gallons on C.

If 18·8 inches be divided into 100 equal parts, the diameter measured by this scale would be 2·13, which, multiplied by itself, gives 4·5369 imperial gallons for the content.

2. Required the content, in imperial gallons, of a sector of a circle, of which the radius is 42 inches, and the arc 118 inches. Ans. 8·9369 imperial gallons.

3. Required the content, in hard soap, of a trapeze, the diagonal being 32 inches, and the perpendiculars upon it from the angles 18 and 14 inches. Ans. 18·865 lbs.

4. Required the content of a quadrant, at 1 inch deep, in plate glass, the radius being 16 inches. Ans. 21·907 lbs.

IV. Some preparation is often necessary before the question can be wrought by the sliding-rule, as in the following examples.

1. Required the content, in imperial gallons, of a segment of a circle, the diameter 50, and the versed sine 10 inches.

Ans. $10\cdot000 \div 50 = \cdot200$ the tabular versed sine, opposite to which is ·1118238 the tabular area; and $\cdot1118238 \times 50^2 \times \cdot0036065 = 1\cdot00823$ imperial gallon.

Set 16·65 on D to 1118 on C; and at 50 on D is 1·01 imperial gallon on C.

2. Required the content, at 1 inch deep, in tallow, of a triangular vessel, of which the sides are 36, 24, and 20 inches.

Here the half sum is 40, and the remainders are 20, 16, and 4. A mean proportional between 20 and 40 is 28·284, and between 16 and 4 is 8. Then,

Set 30·28 on A to 28·284 on B; and against 5 on A is 7·47 lbs. tallow on B.

3. What is the content, in imperial gallons, of an ellipse, of which the axes are 72 and 50 inches?

Ans. 10·17936 imperial gallons.

Prob. X. To gauge solids.

When the depth is greater than one inch, set the gauge-point to the depth instead of 1.

1. Required the content, in imperial gallons, of a rectangular prism, of which the length is 81, the breadth 26, and the depth 25 inches.

Ans. $81 \times 26 \times 25 \times \cdot0036065 = 189\cdot882$ imp. gallons.

By the Gauge-Points.

Set 25 on C to 25 on D; and at 81 on C is 45 on D, a mean proportional between 25 and 81. Then,

Set 16·65 on D to 26 on C; and at 45 on D is 190 imperial gallons on C.

2. Required the content, in imperial gallons and bushels, of an octagonal prism, of which the depth is 80 inches, and each side of the base 63 inches.

Ans. $63^2 \times 80 \times \cdot0174139 = 5529\cdot262$ imperial gallons, $= 691\cdot158$ bushels.

By the Sliding-Rule.

Set 7·578 on D to 80 on C; and at 63 on D is 5529·3 imperial gallons on C.

Set 21·434 on D to 80 on C; and at 63 on D is 691·16 imperial bushels on C.

3. Required the content, in imperial gallons, of a cylindrical vessel, the depth 40 inches, and the diameter of the base 27 inches.

Ans. $27^2 \times 40 \times \cdot0028326 = 82\cdot599$ imperial gallons.

By the Sliding-Rule.

Set 18·8 on D to 40 on C; and at 27 on D is 82·6 imperial gallons on C.

4. Required the content, in imperial gallons, of the frustum of a square pyramid, the depth 24 inches, each side of the lower base 26, and of the higher 34 inches.

Ans. $34 + 26 = 60$, and $(60^2 - 34 \times 26) \times 24 \times \cdot0012022 = 78\cdot364$ imperial gallons.

By the Sliding-Rule.

First set 26 on C to 26 on D; and at 34 on C is 29·72 on D, the mean proportional between 26 and 34. Then,

Set 28·84 on D to 24 on C; and at 60·0 on D is 104·2 o
. . 28·84 . . 24 . . . . . 29·7 . . —25·8 .

78·4 in

NOTE. These, with most other questions, may be wr more easily by Prob. XII. of MENSURATION OF So and therefore it is proper to give tables for it, and the ru working it by the sliding-rule.

TABLE IV.

GAUGE-POINTS TO BE USED WHEN THE MIDDLE ARE TAKEN.

Measures.	Gauge-Points.	
	For Squares.	For Circ
For inches, ...	1·	2·76
Imperial gallons, . .	40·7878	46·02
Imperial bushels, . .	115·3653	130·17
Soft soap, pounds, . .	12·4105	14·00
White soft soap, do. .	12·3839	13·97
Hard soap, do. . .	12·7609	14·3
Tallow, do. . . .	13·4789	15·2
Starch, do. . . .	14·4499	16·3
Green glass, do. . .	7·1246	8·0
Plate glass, do. . .	7·4208	8·3
Broad glass, do. . .	7·9433	8·9
OLD MEASURES.		
Wine gallons, . .	37·2290	42·
Ale gallons, ...	41·1339	46·
Corn gallons, . .	40·1597	45·
Malt bushels, . .	113·5892	128·
Scotch pints, . . .	25·0044	28·
Wheat firlots, . . .	115·2646	130·
Barley firlots, . .	139·2187	157·
Irish gallons, ...	40·3423	45·
Irish barrels, ...	228·2104	257·

Rule for working by the Pen.

Find the squares or products of the sides or diame the top and bottom, and of the double of those in the m the sixth part of the sum of these, multiplied by the multiplier in Table I. or II. will give the content.

By the Sliding-Rule.

Set the gauge-point on D to the length on C; then oppo-e to the sides or diameters at the ends, and to twice that in e middle on D, will be found three numbers on C; and ese three, added together, will give the content.

To work the last question by this rule. $\frac{1}{6}(26^2+34^2+60^2)$ $\times 24\times\cdot 0036065 = 78\cdot 362$ imperial gallons.

By the Sliding-Rule.

Set 40·79 on D to 24 on C; then at 60 on D is 51·6 on C.
. . 40·79 . . 24 . . . . . 34 . . 16·6 . .
. . 40·79 . . 24 . . . . . 26 . . 9·65 . .

77·85 imp. gal.

5. Required the content, in imperial gallons, of a frustum a rectangular pyramid, the depth of the frustum 100 inches, e sides of the upper base 18 and 8 inches, and the sides of e lower base 27 and 12 inches.

Ans. $\frac{1}{6}(18\times 8+27\times 12+45\times 20)\times 100\times\cdot 0036065 =$ ·2282 imperial gallons.

By the Sliding-Rule.

et 40·79 on D to 100 on C; then at 18 on D is 19·47 on C.
. 40·79 . . 100 . . . . . 12 . . 8·65 . .
. 40·79 . . 100 . . . . . 30 . . 54·09 . .

82·21 imp. gal.

6. Required the content, in imperial gallons, of the frustum of a cone, the depth of the frustum 100 inches, and the diameters of the bases 18 and 12 inches.

Ans. $\frac{1}{6}(18^2+12^2+30^2)\times 100\times\cdot 0028326 = 64\cdot 58328$ mperial gallons.

let 46·02 on D to 100 on C; then at 18 on D is 15·3 on C.
. 46·02 . . 100 . . . . . 12 . . 6·8 . .
. 46·02 . . 100 . . . . . 30 . . 42·5 . .

64·6 imp. gal.

7. If the axis of a globe be 100 inches, how many imperial allons will it contain?

In a sphere, the square of twice the middle diameter is three mes the square of the axis.

Ans. $\frac{1}{6}(10000+30000+0)\times 100\times\cdot 0028326$. 1888·4 nperial gallons.

Set 46·02 on D to 100 on C; then at 200 on D is 1888·7 nperial gallons on C.

8. Required the content, in imperial gallons, of a bowl or

segment of a sphere, the depth 15 inches, the diameter of the base 60 inches, and the middle diameter 45 inches.

Ans. $\frac{1}{6}(60^2+90^2+0)\times 15\times \cdot 0028326 = 82\cdot 85355$ imperial gallons.

Set	46·02	on D to	15	on C; then at	60	on D is	25·5	on C.
..	46·02	..	15	....	90	..	57·37	..
							82·87 imp. gl.	

Or by the Rule in Prob. XV. Case 2, of Mensuration of Solids.

Set	32·544	on D to	15	on C; then at	15	on D is	3·18	on C.
..	32·544	..	15	....	30	..	12·75	..
..	32·544	..	15	....	30	..	12·75	..
..	32·544	..	15	....	30	..	12·75	..
							41·43	
							2	
							82·86 imp. gl.	

9. Required the content, in imperial bushels, of a hexagonal prism, of which the depth is 96 inches, and each side of the base 18 inches. Ans. 36·47255 imperial bushels.

10. Required the content, in imperial gallons, of a cylindrical vessel, of which the depth is 84 inches, and the diameter of the base 63 inches. Ans. 944·3775 imperial gallons.

11. Required the content, in pounds of hard soap, of a frustum of a pentagonal pyramid, the depth 60 inches, and the sides of the bases 18 and 6 inches. Ans. 593·355672 lbs.

12. Required the content of the frustum of a cone, in imperial gallons, the depth being 50 inches, and the diameters of the bases 24 and 30 inches. Ans. 103·67316 gallons.

Prob. XI. To gauge malt.

Rule. Take the depths at a great number of places, particularly where the malt is deepest, and where it is ebbest. Add all these depths, and divide the sum by the number of them for a mean depth. Find the content at one inch deep, as before, and multiply it by the mean depth.

1. Required the content of a rectangular floor of malt, of which the length is 72 inches, the breadth 48, and the depth, taken at five different places, 4·7, 5·4, 5·6, 4·9, and 4·4 inches.

Ans. The sum of the depths, 25, divided by 5, gives 5 the mean; then $72\times 48\times 5\times \cdot 0004508 = 7\cdot 7898$ imp. bushels.

By the Sliding-Rule.

Set the length 72 on B to the breadth 48 on MD; then against the depth 5 on A is 7·79 imperial bushels on B.

2. Suppose the length 270, the breadth 56·2, and the mean depth 5·2 inches. Required the quantity of malt.

Ans. $270 \times 56{\cdot}2 \times 5{\cdot}2 \times {\cdot}0004508 = 35{\cdot}570284$ imp. bush.

Set 270 on B to 56·2 on MD; and at 5·2 on A is 35·57 malt bushels on B.

Or find a mean proportional 123·2, between 270 and 56·2. Set 47·097 on D to 5·2 on C; then at 123·2 on D is 35·57 malt bushels on C.

3. Let the length be 140, the breadth 72, and the mean depth 18·2 inches. Required the quantity.

Ans. 82·7 imperial bushels.

4. Let the length be 1250, the breadth 360, and the mean depth 9 inches. Required the quantity.

Ans. 1825·74 imperial bushels.

5. How many imperial bushels of malt are in an octagonal cistern, the length of the side being 10 feet, and the depth in eight different places 10·2, 9·6, 9·1, 9·8, 10·5, 10·7, 10·3, and 10·4 inches? Ans. 24·83 imperial bushels.

6. There is an oval cistern of malt, of which the diameters are 72 and 48, and the depth 5 inches. Required its content.

Ans. 6·118848 imperial bushels.

Find a mean proportional 58·8, between 72 and 48.

Set 53·144 on D to 5 on C; then at 58·8 on D is 6·12 malt bushels on C.

NOTE. Malt must be gauged several times. It is supposed to increase one-fifth in bulk in the cistern, and, after being 40 hours in the heap or floor, it is doubled by sprouting. Therefore, to obtain the net measure, multiply it by ·8 when gauged in the couch, and by ·5 when in the floor. When the measure of the dry barley is given, multiply it by 1·2 to find what it should be in the couch, and that again by 1·6 to find what it should be in the floor.

7. What should be the couch and floor measure of 13·8 imperial bushels of dry barley?

Ans. $13{\cdot}8 \times 1{\cdot}2 = 16{\cdot}56$ the couch measure.

$16{\cdot}56 \times 1{\cdot}6 = 26{\cdot}496$ the floor measure.

8. Suppose a floor to measure 100·8 imperial bushels, what should have been the couch measure? Ans. 63 bushels.

9. Suppose a couch to measure 56 bushels, what should have been the floor measure, and the quantity of dry barley?

Ans. 89·6 the floor measure, and 46⅔ bushels dry barley.

segment of a sphere, the depth 15 inches, the diameter of the base 60 inches, and the middle diameter 45 inches.

Ans. $\frac{1}{6}(60^2+90^2+0)\times 15\times \cdot 0028326 = 82\cdot 85355$ imperial gallons.

Set 46·02 on D to 15 on C; then at 60 on D is 25·5 on C.
. . 46·02 . . 15 . . . . . 90 . . 57·37 . .

82·87 imp. gal.

Or by the Rule in Prob. XV. Case 2, of MENSURATION OF SOLIDS.

Set 32·544 on D to 15 on C; then at 15 on D is 3·18 on C.
. . 32·544 . . 15 . . . . . 30 . . 12·75 . .
. . 32·544 . . 15 . . . . . 30 . . 12·75 . .
. . 32·544 . . 15 . . . . . 30 . . 12·75 . .

41·43
2

82·86 imp. gal.

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By the Sliding-Rule.

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2. Suppose the length 270, the breadth 56·2, and the mean depth 5·2 inches. Required the quantity of malt.

Ans. $270 \times 56{\cdot}2 \times 5{\cdot}2 \times {\cdot}0004508 = 35{\cdot}570284$ imp. bush.

Set 270 on B to 56·2 on MD; and at 5·2 on A is 35·57 malt bushels on B.

Or find a mean proportional 123·2, between 270 and 56·2.

Set 47·097 on D to 5·2 on C; then at 123·2 on D is 35·57 malt bushels on C.

3. Let the length be 140, the breadth 72, and the mean depth 18·2 inches. Required the quantity.

Ans. 82·7 imperial bushels.

4. Let the length be 1250, the breadth 360, and the mean depth 9 inches. Required the quantity.

Ans. 1825·74 imperial bushels.

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Find a mean proportional 58·8, between 72 and 48.

Set 53·144 on D to 5 on C; then at 58·8 on D is 6·12 malt bushels on C.

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9. Suppose a couch to measure 56 bushels, what should have been the floor measure, and the quantity of dry barley?

Ans. 89·6 the floor measure, and $46\frac{2}{3}$ bushels dry barley.

PROB. XII. To gauge open vessels.

These vessels being in the form of prisms, cylinders, frustums, cylindroids, &c. their contents may be found by the preceding rules. But as they are often large and fixed vessels, their contents are generally required at every inch, or tenth part of an inch, of depth. These contents must therefore be found and placed in a table, so that, by taking the depth of the liquor, the content may be known at once from the table.

When the vessels are prisms or cylinders, find the content at one inch deep; and this doubled, tripled, &c. will give the contents at two, three, &c. inches. If the dimensions at the top and bottom be unequal, divide the difference of corresponding sides or diameters at the bases by the depth, to get the difference at one inch deep; and this difference, added to the bottom diameter if it be less than that at the top, or subtracted from it if greater, will give the side or diameter at one inch deep; and the same difference, added to the side or diameter at one inch deep, or subtracted from it, will give it at two inches deep, and so on.

Having found the dimensions, find the content of each part; and, by adding them, the contents at all the depths will be found.

Generally the dimensions are found only in the middle of every six inches, and the content, being found from these dimensions for one inch deep, is added to itself six times, to get the contents for each of these six inches of depth.

1. Suppose an elliptical vessel to be 6 inches perpendicular depth, the axes at the top 65 and 60, and those at the bottom 110 and 100 inches, all taken parallel to the horizon, the vessel inclining so that it requires 15 gallons to reach to the upper part of the bottom where the axes were taken.

The difference of the two greater axes is 45, which, divided by 6, gives 7·5 inches, the difference for every inch of depth; and in the same manner the difference of the lesser axes for every inch of depth is $6\frac{2}{3}$ inches: consequently, at 1 inch from the bottom, the axes will be 72·5 and 66·7 inches; at 2 inches, 80 and 73·3 inches, and so on. These are placed in the second and third columns of the table; and the particular contents being found and added together regularly, both from the top and the bottom, are placed in the fourth and fifth columns.

In such vessels there is a place marked on the edge of the vessel for the dipping-place; and it is here supposed, that, at

the dipping-place, the wet inches are 2, when the 15 gallons are in the vessel to cover the bottom, and also that there is 1 inch dry at the top when the vessel is full.

TABLE.

Dry Inches.	Length.	Breadth.	Content from Top.	Content from Bottom.	Wet Inches.
1	65·0	60	0·00000	136·51854	8
2	72·5	$66\frac{2}{3}$	12·34542	124·17312	7
3	80·0	$73\frac{1}{3}$	27·47622	109·04232	6
4	87·5	80	45·67567	90·84287	5
5	95·0	$86\frac{2}{3}$	67·22704	69·29150	4
6	102·5	$93\frac{1}{3}$	92·41357	44·10497	3
7	110·0	100	136·51854	15·00000	2

Suppose the wet inches at the dipping-place to be 5; then against 5 wet inches in the column titled *Content from Bottom*, is found 90·84287 imperial gallons for the quantity of liquor in the vessel.

2. The depth of a circular mash-tun is 60, the top-diameter 48, and the bottom-diameter 36 inches, and supposing the content of the *drip or fall* to be 20 imperial gallons. Required the content of each 10 inches from the top, and also the whole content.

Ans. First 10 inches 62·57213, whole content 321·7852 imperial gallons.

3. Suppose the depth of a circular tun to be 80 inches, the top-diameter 50, and the bottom-diameter 30. Required the content of the tun, and also of every 10 inches from the bottom, allowing 10 gallons for the *drip or fall*.

Ans. Whole content 380·00836, content of first 10 inches 37·66211 imperial gallons.

Coolers, &c. are very wide and ebb, and their bottoms uneven; therefore the depths must be taken at various parts, and their sum divided by the number of them, to get a mean depth. Tables are constructed for such vessels, exhibiting the content at every tenth part of an inch in depth. They are made and used in the same way as the last table.

It often happens that the depth taken at the dipping-place differs from the mean depth for which the table was calculated. The difference must be marked on the vessel and in the table, with the sign — when the depth at the dipping-place is greater than the mean depth, or with the sign + if it be less; and this difference must be subtracted or added to get the mean depth, before using the table.

Suppose the mean depth to be 4·89, and that at the dipping-place 5 inches; the difference, 0·11, must always be taken from the wet inches to reduce them to mean ones.

Note. When the wort is gauged hot, one-tenth part is deducted from the content, to find how much there will be when cold; as it has been found that 10 gallons of hot wort measure only about 9 gallons when the wort is cold.

1. The length of a cooler is 120, the breadth 84, and the depth at 10 equidistant places 4·6, 4·5, 4·7, 4·4, 4·2, 4, 3·8, 3·7, 3·5, and 3 inches. How many gallons of hot wort will it contain, and how many gallons will there be when the wort is cold?

Ans. 115·6381 gallons hot, and 104·0743 gallons cold wort.

2. Suppose the length to be 280, the breadth 200, and the mean depth 5·1 inches. Required the content in hot, and also in cold wort.

Ans. 808·99056 gallons hot, and 728·0915 gallons cold.

Prob. XIII. To gauge a copper, still, &c.

If the greatest width be at the top, and the least at the bottom, or the contrary, take diameters perpendicular to one another at both ends, and also exactly in the middle, between the top and bottom. (By the bottom is meant the top of the crown in the bottom.) Then work by Prob. XII. of Mensuration of Solids: That is,

To 4 times the product of the middle diameters, add the products of those at the top and of those at the bottom. Multiply the sum by the depth from the top of the vessel to the top of the crown: the product, multiplied by ·0004721, will give the imperial gallons in the content of all above the crown. Water must then be measured into the vessel, just to cover the crown; and this measure, added to that above, will give the whole content.

1. Let the depth to the top of the crown be 36 inches, the diameters at the top 116 and 115·5, at the top of the crown 111 and 110, and in the middle 114 and 113, and the liquor required to cover the crown 16·3 imperial gallons. Required the content.

Ans. $(4 \times 114 \times 113 + 116 \times 115{\cdot}5 + 111 \times 110) \times 36 \times$ ·0004721 $= 1310{\cdot}9726$, and $1310{\cdot}9726 + 16{\cdot}3$ the content of the crown $= 1327{\cdot}2726$ imperial gallons whole content.

If the broadest part be not at the top or bottom; suppose the vessel to be divided into two or more frustums, so that the broadest part of each frustum be at one end of it, and the least breadth at the other. Find the content of each frus

separately, and add these contents, and the liquor required to cover the crown: the sum will be the content.

2. Suppose the depth 36 inches, and the greatest bulge 15 inches from the top; the diameters at the top 80·5 and 80·8, and at the bulge 89·0 and 89·5, and in the middle between these 85·5 and 86·0; also, the diameters at the top of the crown 83·0 and 83·5, and half-way between it and the greatest bulge 86·5 and 87·0; the liquor required to cover the crown 18·5 gallons. Required the content.

Ans. 775·36435 imperial gallons.

3. Let the depth of a still be 42·8 inches, and the height of the greatest bulge from the bottom 20·5; and let the diameters at the top be 21·0, and at the bulge 47·8 and 47·3, and half-way between them 45·4 and 46·0; also, the diameters at the bottom 43·5 and 44·0, and half-way between the bottom and the bulge 47·0 inches. Required the content in imperial gallons, supposing 7 gallons to cover the crown.

Ans. 249·31137 imperial gallons.

Stills are generally measured by taking cross diameters at the middle of every six inches, and finding the content of each part as if it were a cylinder; and the top is calculated like a frustum or zone of a sphere.

4. Suppose the top of the still to be 7·3 inches, and its greatest diameters 41·5 and 40·8, and its least 21·0; the body of the still 35·5 inches deep, and the cross diameters in the middle of every six inches from the top to be, first, 43·9 and 43·2; second, 47·0 and 46·2; third, 47·8 and 47·3; fourth, 47·6 and 47·4; fifth, 46·5 and 46·5; and in the middle of the undermost $5\frac{1}{2}$ inches, 45·0 and 45·2 inches. Required the content in imperial gallons, supposing 7·5 gallons to cover the crown. Ans. 248·226245 imperial gallons.

CASK GAUGING.

The easiest way of finding the contents of casks is by the diagonal-rod.

OF THE DIAGONAL-ROD.

This rod is 4 feet long and ·4 of an inch square. It is divided into 4 equal parts by joints. The principal line on it is the diagonal line for imperial gallons, which may be made thus:

It is found by experiment, that a cask containing 144 imperial gallons has a diagonal of 40 inches: therefore 144 is placed at 40 inches; and, since the contents are as the cubes of the diagonals, $144 : 40^3 = 64000 :: 114 : 152000$, the cube root of which is 37, therefore 114 is put at 37; and

in the same manner any other number of gallons may be placed upon the rod. A line of inches is also upon the same side of the rod.

Upon another side of the rod is a line marked Seg. St. for finding the ullage of a standing cask.

On a third side are tables for ullaging lying casks, viz. those of half or whole hogsheads, of 84, 108, 110, and 120 old wine gallons. The depth is taken in inches, and the ullage is given in gallons.

The fourth side contains lines for ullaging casks of known dimensions, as a half-anker, a firkin, a barrel, a hogshead, a puncheon, &c. either lying or standing. Put into the bung that end of the rod from which the divisions for the given cask are numbered, until it rests upon the opposite stave; and the division on the rod intersected by the surface will be the ullage.

Prob. XIV. To find the content of a cask by the rod.

Put in the end covered with brass at the bung, and extend it to the opposite corner of the head, and mark the gallons and parts at the middle of the bung; then extend it to the other head of the cask, and mark the gallons and parts. Half the sum of these two, if they do not agree, will be the content.

Note. The contents on the rod are made for the most common forms of casks.

1. Suppose a cask to be 21 inches long, the bung-diameter 19, and the head-diameter 16. Required the content in ale gallons.

If the rod be extended from the bung to the opposite corner of the head, it will give 19·3 imperial gallons nearly.

Prob. XV. To find the content of a cask by the pen.

In common casks, the cube of the diagonal divided by $444\frac{4}{9}$ will give the content in imperial gallons. Therefore, to the square of half the length add the square of half the sum of the diameters, to get the square of the diagonal: this multiplied by its square root, and divided by $444\frac{4}{9}$, gives the content.

Half the sum of the diameters in last example is 17·5; therefore $17{\cdot}5^2 + 10{\cdot}5^2 = 416{\cdot}5$, the square root of which is 20·4, and $416{\cdot}5 \times 20{\cdot}4 \div 444\frac{4}{9} = 19{\cdot}12$ imperial gallons the content.

OF THE VARIETIES OF CASKS.

Casks are commonly divided into four varieties, according to the degree of their curvature.

i. The middle zone of a spheroid, measured by Prob. XX. of MENSURATION OF SOLIDS.

ii. The middle zone of a parabolic spindle, gauged by Prob. XXX. of the same.

iii. Two equal frustums of a parabolic conoid, by Prob. XXIV. of the same.

iv. Two equal frustums of a cone, by Prob. X. of the same.

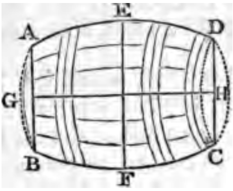
2. Required the content, in imperial gallons, of a cask of the first variety, of which the length is 40, the bung-diameter 32, and the head-diameter 24 inches.

$$(2 \times 32^2 + 24^2) \times 40 = 104960$$
$$\cdot 0009442$$
$$\overline{99 \cdot 1032} \text{ imperial gallons.}$$

By the Sliding-Rule.

Set 32·544 on D to 40 on C; and at 24 on D is 21·75 on C.
. . 32·544 . . 40 . . . . . 32 . . 38·67 . .
. . 32·544 . . 40 . . . . . 32 . . 38·67 . .

99·09 imp. gals.



3. Suppose the cask to be of the second variety, and the dimensions the same as in the last.

$$(2 \times 32^2 + 24^2 - \cdot 4 \times 8^2) \times 40 = 103936$$
$$\cdot 0009442$$
$$\overline{98 \cdot 1364} \text{ imperial gallons.}$$

Set 32·544 on D to 40 on C; then at 8 on D is 2·417 on C, which, multiplied by ·4, gives ·9668 of an imperial gallon to be taken from the content found in the last example, and leaves 98·13643 imperial gallons.

4. Let the cask be of the third variety, and the dimensions as before.

Ans. $(32^2 + 24^2) \times 20 \times \cdot 0028326 = 90 \cdot 6432$ imp. gals.

Set 18·95 on D to 20 on C; then at 32 on D is 58·01 on C.
. . 18·95 . . 20 . . . . . 24 . . 32·63 . .

90·64 imp. gal.

5. Let the cask be of the fourth variety, and the dimensions still the same.

Ans. $(56^2 - 32 \times 24) \times 40 \times \cdot 0009442 = 89 \cdot 4346$ imperial gallons.

Set 24 on D to 24 on C; and at 32 on C is 27·7 on D, the mean proportional.

Set 29·7 on D to 40 on C; and at 56 on D is 118·4 on C.
. . 29·7 . . 40 . . . . . 27·7 . . 29·0 . .

89·4 imp. gal.

6. Let the length be 20, and the diameters 16 and 12 inches. Required the contents in imperial gallons, according to all the varieties.

Ans. First var. 12·3879, second var. 12·267, third var. 11·3304, fourth var. 11·1793 imperial gallons.

7. Let the length be 40, and the diameters 32 and 26 inches. Required the content, according to all the varieties.

Ans. First var. 102·88, second var. 102·3362, third var. 96·3084, fourth var. 95·6286 imperial gallons.

8. Let the length be 45 inches, and the diameters 36 and 30 inches. Required the content, according to all the varieties, in imperial gallons.

Ans. First var. 148·3716, second var. 147·7597, third var. 139·9588, fourth var. 139·194 imperial gallons.

9. Let the length be 48 inches, and the diameters 40 and 32 inches. Required the content, according to all the varieties.

Ans. First var. 191·4384, second var. 190·2782, third var. 178·3858, fourth var. 176·9355 imperial gallons.

NOTE. The second variety comes nearer to the form of common casks than any of the others, but it does not entirely agree with them.

PROB. XVI. To gauge a cask by reducing it to a cylinder.

RULE. Divide the head by the bung diameter, and find the quotient in the column titled *Quot.* in the following table. In the column answering to the variety of the cask, on the same line with the quotient, will be found a number, which, multiplied by the difference between the bung and head diameters, and the product added to the head diameter, will give the mean diameter, or that of a cylinder equal to the cask. Then multiply the square of the mean diameter by the length of the cask, and by ·0028326, for the content in imperial gallons.

By the Sliding-Rule.

Find the difference between the head and bung diameters on the edge of the rule, and against it, in the proper line, is the number to be added to the head, to get the diameter of the cylinder, called the mean diameter.

Quot.	1st Var.	2d Var.	3d Var.	4th Var.	Quot.	1st Var.	2d Var.	3d Var.	4th Var.
·50	·732	·693	·581	·527	·76	·695	·678	·534	·511
·51	·730	·692	·579	·527	·77	·694	·677	·532	·510
·52	·729	·692	·577	·526	·78	·693	·677	·530	·510
·53	·727	·691	·575	·526	·79	·691	·676	·529	·510
·54	·726	·690	·573	·525	·80	·690	·676	·527	·509
·55	·724	·690	·571	·524	·81	·689	·675	·526	·508
·56	·723	·689	·569	·523	·82	·688	·675	·524	·508
·57	·721	·689	·567	·523	·83	·686	·674	·522	·508
·58	·720	·688	·565	·522	·84	·685	·674	·521	·507
·59	·719	·688	·563	·521	·85	·684	·673	·520	·506
·60	·717	·687	·562	·521	·86	·683	·673	·519	·506
·61	·716	·686	·559	·520	·87	·682	·672	·517	·505
·62	·714	·686	·558	·519	·88	·680	·671	·516	·505
·63	·713	·685	·556	·519	·89	·679	·671	·515	·504
·64	·712	·685	·554	·518	·90	·678	·671	·513	·504
·65	·710	·684	·552	·517	·91	·677	·670	·511	·503
·66	·709	·684	·551	·517	·92	·675	·670	·510	·503
·67	·708	·683	·549	·516	·93	·674	·669	·509	·503
·68	·706	·682	·547	·516	·94	·673	·668	·507	·502
·69	·705	·682	·545	·516	·95	·672	·668	·506	·501
·70	·703	·681	·543	·515	·96	·670	·667	·505	·500
·71	·702	·681	·541	·514	·97	·670	·667	·503	·500
·72	·701	·680	·540	·513	·98	·667	·666	·501	·500
·73	·699	·680	·539	·513	·99	·666	·666	·500	·500
·74	·698	·679	·537	·512	1·00	—	—	—	—
·75	·697	·678	·535	·512					

1. Suppose the length 40, and the diameters 32 and 26 inches. Required the mean diameter and content in imperial gallons, according to all the varieties.

$26 \div 32 = ·81$, opposite to which, in the table, are ·689, ·675, ·526, ·508. Then,

$6 \times ·689 + 26 = 30·134$ mean diameter, and $30·134^2 \times 40 \times ·0028326 = 102·8866$ imperial gallons in first variety.

$6 \times ·675 + 26 = 30·05$ mean diameter, and $30·05^2 \times 40 \times ·0028326 = 102·3138$ imperial gallons in second variety.

$6 \times \cdot526 + 26 = 29\cdot156$ mean diameter, and $29\cdot156^2 \times 40 \times \cdot0028326 = 96\cdot3166$ imperial gallons in third variety.

$6 \times \cdot508 + 26 = 29\cdot048$ mean diameter, and $29\cdot048^2 \times 40 \times \cdot0028326 = 95\cdot6044$ imperial gallons in fourth variety.

Set 18·79 on D to 40 on C; and at 30·188 on D is 103·96 imperial gallons on C, first variety.

Set 18·79 on D to 40 on C; and at 30·05 on D is 102·31 imperial gallons on C, second variety.

Set 18·79 on D to 40 on C; and at 29·156 on D is 96·3 imperial gallons on C, third variety.

Set 18·79 on D to 40 on C; and at 29·048 on D is 95·6 imperial gallons on C, fourth variety.

2. Suppose the length 60, and the diameters 40 and [illegible] inches. Required the content, according to all the varieties.

Ans. First var. 239·25563, second var. 237·82937, third var. 222·91406, fourth var. 221·14491 imperial gallons.

3. Suppose the length 50, and the diameters 36 and [illegible] inches. Required the content, according to all the varieties.

Ans. First var. 164·84336, second var. 164·1483, third var. 155·47142, fourth var. 154·68408 imperial gallons.

4. Suppose the length 56, and the diameters 40 and [illegible] inches. Required the content, according to all the varieties.

Ans. First var. 237·71933, second var. 237·37557, third var. 229·68268, fourth var. 229·2483 imperial gallons.

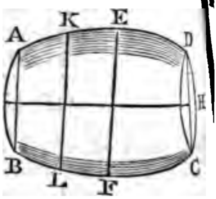
Prob. XVII. To gauge a cask by the middle diameter.

Add the squares of the head, of the bung, and of twice the middle diameter: the sum, multiplied by the length, and by the proper multiplier, gives the content.

Note. This is the most accurate method of finding the contents of casks.

1. Let the length of the cask be 40, and the diameters 32 at the bung, 26 at the head, and 30·4 inches in the middle.

Ans. $(32^2 + 26^2 + 60\cdot8^2) \times 40 \times \cdot0004721 = 101\cdot91015$ imperial gallons.



Set 46·024 on D	to 40 on C;	and at 60·8 on D	is 69·81 on C.	
. . 46·024	. . 40	. . . . . 32·0	. . 19·34	. .
. . 46·024	. . 40	. . . . . 26·0	. . 12·76	. .
			101·91 imp. g	

2. Let the length be 42, the bung diameter 34, the head 27, and the middle diameter 32 inches. Required the content in imperial gallons. Ans. 118·5925 imperial gallons.

3. Let the length be 44, the head 30, the bung 36, and the middle diameter 33 inches. Required the content.

Ans. 136·1008 imperial gallons.

4. Let the length be 50, the middle 36, the head 34, and the bung diameter 40 inches. Required the content.

Ans. 187·4237 imperial gallons.

PROB. XVIII. To find the content of a cask without the middle diameter.

RULE. From 12 times the head subtract 7 times the bung diameter, and multiply the remainder by twice the bung diameter, and subtract the product from the square of 5 times the sum of these diameters. Multiply the remainder by the length, and by ·00003147: the product will give the content in imperial gallons.

1. Let the length of the cask be 40 inches, the bung diameter 32, and the head diameter 24 inches. Required the content.

Ans. $(12 \times 24 - 7 \times 32) \times 64 = (288 - 224) \times 64 = 64 \times 64 = 4096$ and $5 \times (32 + 24) = 280$, and $280^2 - 4096 = 74304$ and $74304 \times 40 \times \cdot 00003147 = 93 \cdot 534$ imp. gallons.

2. Suppose the length to be 41 inches, the bung diameter 32·2, and the head diameter 26·3 inches. Required the content. Ans. 102·89564 imperial gallons.

3. Let the length be 45, the bung 34, and the head diameter 28 inches. Required the content.

Ans. 126·2299 imperial gallons.

4. Let the length be 48, the bung 36, and the head diameter 30 inches. Required the content.

Ans. 152·75387 imperial gallons.

OF ULLAGING CASKS.

THE Ullage of a cask is the quantity of liquor in it when it is not full. The dimensions are taken either when it is lying on its side, or when it is standing on its end. The depth of the liquor is called the Wet Inches, and the remainder the Dry Inches.

PROB. XIX. To find the ullage of a standing cask by the pen.

Add together the squares of the diameter at the top of the liquor, of the diameter at the nearest end, and of twice the diameter half-way between these two, and multiply the sum by the length or distance from the surface of the liquor to the nearest end, and by ·0004721: the product will be the content of the lesser part of the cask in imperial gallons, whether full or empty.

1. Suppose the wet inches to be 10, and the diameters 24, 27, and 29 inches. Required the ullage.

Ans. $(24^2+29^2+54^2)\times 10\times{\cdot}0004721=20{\cdot}4561$ imperial gallons.

2. Suppose the length 40, the wet inches 30, the diameters 30, 24, and 28 inches, bung diameter 32, and the middle $30\frac{1}{2}$. Required the ullage.

$(24^2+30^2+56^2)\times 10\times{\cdot}0004721=\ 21{\cdot}773$ im. gal. empty.
$(24^2+32^2+61^2)\times 40\times{\cdot}0004721=100{\cdot}482$ . . cask.

78·709 . . full.

3. Suppose the length 28, the wet inches 12, the diameters 20, 22, and 24 inches, bung diameter 25, and the middle 23 inches. Required the ullage. Ans. 16·4971 imperial gallons.

4. Suppose the length 50, the wet inches 30, the diameters 26, 30, and 32 inches, bung diameter 36 inches, and the middle 32 inches. Required the ullage.

Ans. 93·1925 imperial gallons.

Otherwise. Multiply the square of the dry or wet inches (the greater of the two) by the difference between the head and bung diameters, and divide the product by the square of the length: the quotient, subtracted from the bung diameter, will give the mean diameter.

Multiply the square of the mean diameter by the wet or dry inches (the lesser of the two), and then by the proper multiplier, to get the content of the filled or empty part (the lesser of the two).

5. Suppose the length 40, the bung 32, the head diameter 26, and the wet inches 10. Required the ullage.

Ans. $30^2\times 6\div 1600=3\frac{3}{8}$, and $32-3\frac{3}{8}=28\frac{5}{8}$, and $(28\frac{5}{8})^2\times 10\times{\cdot}0028326=23{\cdot}21006$ imperial gallons.

6. Suppose the length 60, the bung 49, the head diameter 40, and the wet inches 25. Required the ullage.

Ans. 149·4376 imperial gallons.

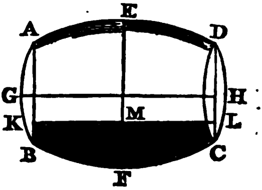
. Suppose the length 48, the bung 40, the head diameter and the wet inches 18. Required the ullage.

Ans. 72·2989 imperial gallons.

ᵖROB. XX. To ullage a lying cask by the pen.

Divide the wet or dry inches (the least of the two) by the g diameter, and find the quotient in the column of versed s in the table of segments. Take out its corresponding nent, and multiply it by the content of the cask, and by the product is the ullage in gallons.

. Suppose the content 92 imial gallons, the bung diameter and the wet inches 8.



$2)8{\cdot}00({\cdot}25$ the versed sine, of ich the segment is ·153546, and $3546 \times 92 \times 1\frac{1}{4} = 17{\cdot}65779$ erial gallons the ullage.

:. Let a lying cask be 40 es long, and the diameters 32, and $30\frac{1}{2}$. Required the ullage, when the dry inches are

Ans. 67·6465 imperial gallons.

– Let a lying cask be 46 inches long, and the diameters 30, and 28. Required the ullage, when the wet inches 8·5 and 23 inches respectively.

Ans. 23·0914 and 86·95 imperial gallons.

- Let a lying cask be 56 inches long, and the diameters 34, and 36. Required the ullage, when the dry inches 30, 18, and 12 respectively.

Ans. 40·2894, 132·9365, and 157·9166 imperial gallons.

therwise. Find a mean diameter according to the va- of the cask; from the wet inches subtract half the differ- between the bung and mean diameter, divide the re- der by the mean diameter, and find the quotient in the mn of versed sines in the table of segments. Take out its esponding segment, and multiply it by the square of the n diameter, by the length of the cask, and by ·0036065: product will be the ullage in imperial gallons.

. Let the length of a lying cask of the first variety be 40 es, the bung diameter 30, the head diameter 24, and the inches 12. Required the ullage.

$4 \div 30 = {\cdot}80$, opposite to which in the table, page 237, 590, and $\cdot 690 \times 6 + 24 = 28{\cdot}14$ mean diameter; then $-{\cdot}93) \div 28{\cdot}14 = {\cdot}393\frac{1}{4}\frac{8}{8}\frac{5}{9}$ versed sine, the corresponding nent of which is ·286881, and $\cdot 286881 \times 28{\cdot}14^2 \times 40 \times$ $6065 = 32{\cdot}77147$ imperial gallons the ullage.

X

2. Let the length of a lying cask of the first variety be 48 inches, the bung diameter 32, the head diameter 24, and the wet inches 14. Required the ullage.

Ans. 49·2603 imperial gallons.

3. Let the length of a lying cask of the second variety be 38 inches, the bung diameter 36, the head diameter 32, and the wet inches 18. Required the ullage.

Ans. 64·74223 imperial gallons.

4. Let the length of a lying cask of the third variety be 50 inches, the bung diameter 45, the head diameter 36, and the wet inches 15. Required the ullage.

Ans. 63·74105 imperial gallons.

Prob. XXI. To ullage a cask by the sliding-rule.

First find the whole content of the cask. Next set the length or bung diameter on the slider to 100 on the rule, and against the wet or dry inches on the slider is a number upon Seg. St. or upon Seg. Ly. to be reserved. Then set 100 on B to this reserved number on A; and opposite to the content on B will be found the ullage on A.

1. Suppose the length of a standing cask 40 inches, the wet inches 10, and the content 92 imperial gallons.

Set 40 on the slider to 100 on the rule; and at 10 on the slider is 23 on Seg. St. to be reserved.

Set 100 on B to 23 on A; and at 92 on B is 21·2 imperial gallons on A.

2. Let the bung diameter of a lying cask be 32 inches, the wet inches 8, and the content 92 gallons. Required the quantity of liquor in it. Ans. 16·4 gallons.

3. Let the length of a standing cask be 20 inches, its content 11·5 gallons, and the wet inches 5. Required the ullage.

Ans. 2·65 gallons.

4. Let the diameter of a lying cask be 34 inches, the wet inches 25, and the content 138 gallons. Required the ullage.

Ans. 28·6 gallons dry.

SPECIFIC GRAVITY.

ight of a cubic foot of a body, in proportion to that of 'oot of water, is called its Specific Gravity.
ic foot of water, at the temperature of 40° of Fahren- ermometer, weighs 1000 ounces avoirdupois; and : the following table of specific gravities expresses in he weight of a cubic foot of these bodies.

TABLE OF SPECIFIC GRAVITIES.

SOLIDS.

from 16000 to	23000	Spar, heavy, . . .	4430
l, hammered,	19326	Jargon of Ceylon, .	4416
f George III.	17629	Ruby, oriental, .	4283
, . .	17600	Garnet, precious, .	4230
at 32° Fahr.	13598	. . . . common, . .	3576
. .	11352	Topaz, . from 3536 to	4061
n, . .	11800	Sapphire, oriental, .	3994
. .	11000	Diamond, from 3523 to	3550
er, . .	10744	Beryl, oriental, . .	3549
of George III.	10534	Granite, ...	3500
molten, .	9823	English flint glass, .	3329
f Japan, .	9000	Tourmaline, .	3155
ire-drawn, .	8878	Hornblende, . .	3000
ed, molten, .	8788	Asbestus, . .	2996
1, . .	8694	Limestone, . .	2950
na, . .	8611	Basalt, . .	2860
ire-drawn, .	8544	Marble, Parian, .	2837
mmon, . .	7824	. . . . . green Campanian,	2742
. . .	8306	. . . . Egyptian,	2668
nolten, .	8279	Chalk, British, .	2784
, . .	8109	Emerald of Peru, .	2775
iron, hammered,	7956	Jasper, . .	2710
. .	7833	Glass, white, . .	2892
nolten, .	7812	. . . . bottle, . .	2733
, ...	7788	. . . . green, .	2642
1, Carron, .	7248	Pearl, oriental, .	2684
hammered, .	7787	Coral, . .	2680
. .	7471	Slate, . . .	2670
dened, .	7299	Rock crystal, .	2653
re Cornish, .	7291	Quartz, . .	2640
olten, . .	7191	Pebble, English, .	2619
n, ...	7119	Felspar, . .	2564
ny, . .	6702	Stone, common, .	2500
m, . .	6115	Porcelain, China, .	2385
m, . .	5900	. . . . . . Limoges, .	2341
f the globe, .	5200	Obsidian, . .	2348
ne, . .	4930	Gypsum, . .	2280

Clay,	2160	Butter,	942
Opal,	2114	Ice,	930
Sulphur, native,	2033	Gunpowder, shaken,	922
Brick,	2000	Pumice-stone,	915
Ivory,	1917	Logwood,	913
Nitre,	1900	Living men,	891
Alabaster,	1874	Potassium,	866
Gunpowder, solid,	1745	Beech,	852
Alum,	1714	Ash,	845
Bone, dry,	1660	Apple-tree,	793
Sand,	1500	Maple,	755
Gum Arabic,	1452	Citron,	726
Opium,	1337	Orange-tree,	705
Ebony, American,	1331	Walnut,	681
Lignumvitæ,	1327	Pear-tree,	661
Coal,	1250	Hazel,	609
Pitch,	1150	Linden-tree,	604
Rosin,	1100	Elm,	600
Amber,	1078	Cypress,	598
Mahogany,	1063	Cedar, American,	561
Brazil-wood, red,	1031	Fir, male,	550
Boxwood,	1030	... female,	498
Sodium,	973	Poplar,	383
Oak, heart of,	950	Cork,	240

LIQUIDS.

Sulphuric acid,	1845	Wine of Burgundy,	991
Boracic acid,	1830	... red port,	990
Nitrous acid,	1500	Castor oil,	970
Aqua-fortis,	1500	Linseed oil,	940
Honey,	1450	Proof spirit,	935
Water of the Dead Sea,	1240	Whale oil,	923
Aqua regia,	1234	Moselle wine,	916
Nitric acid,	1218	Olive oil,	915
Muriatic acid,	1170	Muriatic ether,	874
Strong ale, from 1020 to	1050	Oil of turpentine,	870
Human blood,	1045	Brandy,	837
Milk,	1030	Alcohol, absolute,	792
Sea water,	1026	Sulphuric ether,	739
Tar,	1015	Oxygen gas,	1·35
Distilled water,	1000	Air at earth's surface, about	$1\frac{1}{4}$
White Champagne,	997	Azotic gas,	1·2
Wine of Bordeaux,	994	Hydrogen gas,	0·1

PROB. I. To find the magnitude of a body from its weight.

RULE. Divide the weight of the body by its specific gravity, both being in ounces: the quotient is the content in cubic feet.

1. How many cubic inches are in 1 lb. of gunpowder?

Ans. $1728 \times 16 \div 922 = 30$ inches nearly.

2. What is the content of a block of Parian marble weighing 5 cwt.? Ans. 3·158 cubic feet.

3. What is the content of a ton weight of mahogany? Ans. 33·716 cubic feet.

4. What is the content of a block of granite which weighs 4 tons? Ans. 40·96 cubic feet.

5. What is the content of a cast-iron ball which weighs 100 lb.? Ans. 381·457 cubic inches.

PROB. II. To find the weight of a body from its magnitude.

RULE. Multiply the content in feet by the specific gravity: the product is the weight in ounces.

1. What is the weight of a stone of green Campanian marble 63 feet long, and its breadth and thickness each 12 feet?

Ans. $63 \times 12 \times 12 \times 2742 = 24875424$ oz. $= 694\frac{11}{100}$ tons.

2. What is the weight of a log of beech 10 feet long, 3 broad, and $2\frac{1}{2}$ feet thick? Ans. $3993\frac{3}{4}$ lb.

3. What is the weight of a cast-iron ball 2 inches in diameter? Ans. 13·177 ounces.

4. What is the weight of a log of mahogany 40 feet long, 3 broad, and $2\frac{1}{2}$ thick? Ans. 8·898 tons.

5. What is the weight of a leaden ball 6 inches in diameter? Ans. 185·747 ounces.

PROB. III. To find the specific gravity of a body.

CASE I. When the body is heavier than water.

RULE. Weigh the body both in air and in water, and, annexing three ciphers to the weight in air, divide by the difference of the weights, to get the specific gravity.

1. Suppose a piece of stone to weigh 7 lb. in air, and 5 lb. in water. What is its specific gravity?

Ans. $7000 \div 2 = 3500$ ounces the specific gravity.

2. A piece of copper weighs 36 oz. in air, and 32 in water. What is its specific gravity? Ans. 9000.

3. Suppose a piece of gold weighs 40 lb. in air, and 37·93 lb. in water. What is its specific gravity? Ans. 19324 nearly.

4. Suppose a piece of platina weighs 10 lb. in air, and 9·5 lb. in water. What is its specific gravity? Ans. 20000.

CASE II. When the body is lighter than water.

RULE. Having weighed the light body in air, and a body heavier than water both in air and in water, fasten them together with a slender tie, and weigh the compound in water, and subtract it from the weight of the heavy body in water, and to the remainder add the weight of the light body in air,

and by the sum divide the weight of the light body in air with three ciphers annexed: the quotient is the specific gravity of the light body.

1. A piece of copper weighs 18 lb. in air, and 16 lb. in water; a piece of elm which weighs 15 lb. in air is fixed to the copper; and the compound weighs 6 lb. in water. What is the specific gravity of the elm?

Ans. $15000 \div (16 - 6 + 15) = 600$ the specific gravity of the elm.

2. A piece of copper which weighs 27 ounces in air, and 24 in water, is attached to a piece of cork which weighs 6 ounces in air, and the compound weighs 5 ounces in water. What is the specific gravity of the cork? Ans. 240.

3. A piece of lead weighs 60 lb. in air, and 55 lb. in water; a piece of poplar which weighs 30 lb. in air is fixed to the lead; and the compound weighs 7 lb. in water. What is the specific gravity of the poplar? Ans. $383\frac{1}{3}$.

4. A piece of steel weighs 140 lb. in air, and 122 lb. in water; a piece of fir which weighs 30 lb. in air is fixed to the steel; and the compound weighs $97\frac{1}{2}$ lb. in water. What is the specific gravity of the fir? Ans. $550\frac{50}{109}$.

Prob. IV. Given the specific gravity and the weight of a mass composed of two ingredients, and also the specific gravity of each ingredient; to find the quantity of each of them.

Rule. As the specific gravity of the mass, multiplied by the difference between those of the ingredients, is to the specific gravity of the most valuable ingredient, multiplied by the difference between those of the mass and the other ingredient, so is the whole weight to the weight of the highest ingredient; and that of the other may be found in the same way.

1. A composition of 112 lb. is made of copper of Japan and tin. Required the quantity of each ingredient, the specific gravity of the mass being 8784.

Ans. $(9000 - 7291) \times 8784 : (8784 - 7291) \times 9000 ::$ $112 : 100{\cdot}25$ lb. of copper.

2. A mixture of gold and silver weighed 170 lb. and its specific gravity was 15630. Required the quantity of each metal in it. Ans. 119·673 lb. gold, 50·327 lb. silver.

3. A composition of 100 lb. is made of platina and steel, and its specific gravity is 15000. Required the quantity of each ingredient. Ans. 78·54 lb. platina, 21·64 lb. steel.

4. A composition of silver and steel weighs 1000 lb. and its

specific gravity is 8000. Required the quantity of each ingredient. Ans. 77·046 lb. silver, 922·954 lb. steel.

TO FIND THE TONNAGE OF A SHIP.

The length is taken in a straight line along the rabbet of the keel, from the back of the main sternpost to a perpendicular from the fore part of the main stem, under the bowsprit, from which subtract $\frac{3}{5}$ of the breadth: the remainder is the length. The breadth is taken at the broadest part of the ship, from the outside to the outside.

Rule. Multiply the square of the breadth by the length, and divide the product by 188: the quotient will be the tonnage.

1. Required the tonnage of a ship, of which the length is 75 feet, and the breadth 26 feet.

Ans. $26 \times 26 \times 75 \div 188 = 269\frac{32}{47}$ tons.

2. Required the tonnage of a ship, of which the length is 96 feet, and the breadth 33 feet. Ans. $556\frac{4}{47}$ tons.

3. Required the tonnage of a ship, of which the length is 100 feet, and the breadth 40 feet. Ans. $851\frac{3}{47}$ tons.

4. Required the tonnage of a ship, of which the length is 150 feet, and the breadth 60 feet. Ans. $2872\frac{16}{47}$ tons.

Note. This rule is very erroneous, and no other general rule can be given that is perfectly accurate. The best way is to find the quantity of water displaced by the ship when she is loaded; but as this must be done by means of ordinates, the operation is laborious. It is easier to load her with ballast, weighing the load as it is put on board.

The following rule is a near approximation.

1*st*, *For Ships of War.* Take the length of the gun-deck, from the rabbet of the stem to that of the sternpost; subtract $\frac{1}{24}$ of it: the remainder is the length. Take the extreme breadth from outside to outside of the plank, and add it to the length: $\frac{1}{23}$ of the sum is the depth. Set up this height from the limber-strake, and at that height take a breadth from outside to outside, where the extreme breadth was taken, and take another breadth in the middle, between this and the limber-strake: add the extreme and these two breadths, and take $\frac{1}{3}$ of the sum for the breadth. Then multiply the length, breadth, and thickness, and divide the product by 49.

2*d*, *For Ships of Burden.* Take the length of the lower deck, from the rabbet of the stem to that of the sternpost, and from it subtract $\frac{1}{32}$ of it, for the length. Take the extreme breadth from outside to outside, and add it to the length of the lower deck: $\frac{5}{55}$ of the sum is the depth. Set up this

depth from the limber-strake, where the extreme breadth was taken, and at this height take a breadth from outside to outside, take another breadth at $\frac{2}{3}$ of this height, and a third at $\frac{1}{3}$ of the height: add these three to the extreme breadth, and $\frac{1}{4}$ of the sum is the mean breadth. Multiply the length, breadth, and depth, and divide 3 times the product by 110 for the tonnage.

TO FIND THE WEIGHT OF CATTLE.

TAKE the girt behind the shoulder, and the length from the fore part of the shoulder-blade to the buttock, both in feet. Multiply the square of the girt by 4 times the length, and divide by 21: the quotient is the weight, nearly, of the four quarters, in stones of 16 lb., each lb. $17\frac{1}{2}$ ounces avoirdupois.

NOTE. The four quarters are little more than the half of the whole weight; the skin weighs about the 18th part, and the tallow about the 12th part.

1. What is the weight of the four quarters of an ox, of which the girt is 6 feet 6 inches, and the length 5 feet 10 inches? Ans. $6{\cdot}5^2 \times 23\frac{1}{3} \div 21 = 46$ stones $15\frac{1}{3}$ lb.

2. What is the weight of the quarters of a sheep, of which the girt is 3 feet 1 inch, and the length 2 feet 8 inches? Ans. 4 stones 13·26 lb.

3. What is the weight of a hog which is 4 feet 6 inches in girt, and 3 feet 4 inches in length? Ans. $12\frac{6}{7}$ stones.

4. What is the weight and value of an ox measuring $6\frac{1}{2}$ feet in girt, and $5\frac{3}{4}$ feet in length, at 11s. 6d. a stone, sinking offals? Ans. 46·274 stones, value £26·6075.

5. What was the value of the four quarters of the Duncarn ox, which measured 9 feet $3\frac{1}{2}$ inches in girt, and 5 feet $7\frac{1}{2}$ inches in length, at 10s. 6d. a stone? Ans. £48, 11s. 3d.

6. What is the weight of the four quarters of a calf measuring 3 feet in girt by $2\frac{1}{4}$ feet in length? Ans. $3\frac{6}{7}$ stones.

TO FIND THE WEIGHT OF A STACK OF HAY.

To the height from the ground to the eaves, add half the height from the eaves to the top; then multiply the sum, and the length and breadth of the stack, into one product, all of them being taken in feet. Divide the product by 27, to bring it to yards. This, multiplied by 6, will give the number of stones, if the hay be new; but if the stack has stood a considerable time, add a third to it; or if it be old hay, add a half to it.

1. How much hay does a new stack contain, of which the length is 25 feet, the breadth 9 feet, the height from the ground to the eaves 14 feet, and above the eaves 8 feet?

Ans. $18 \times 25 \times 9 \times 6 \div 27 = 900$ stones.

2. How much old hay in a stack 40 feet long and 16 feet broad, the height to the eaves 15 feet, and above 8 feet?

Ans. $4053\frac{1}{3}$ stones.

3. How much new hay in a stack 50 feet long and 30 feet broad, the height to the eaves 20 feet, and above 14 feet?

Ans. 9000 stones.

4. How much hay in a stack which has stood 4 weeks, 60 feet long and 35 feet broad, the height to the eaves 24 feet, and above 16 feet? Ans. $19911\frac{1}{9}$ stones.

OF BALLS AND SHELLS.

An iron ball 4 inches in diameter weighs 9 lb. nearly; and a leaden ball $4\frac{1}{4}$ inches in diameter weighs about 17 lb. Also, a pound of gunpowder measures about 30 cubic inches. And similar solids are to one another as the cubes of their diameters, or like sides.

Prob. I. Given the diameter of an iron ball, to find its weight, and conversely.

Rule. Divide the cube of the diameter by $7\frac{1}{9}$: the quotient will be the weight in pounds.

Multiply the weight by $7\frac{1}{9}$: the cube root of the product is the diameter.

1. What is the weight of an iron ball, of which the diameter is $3\frac{1}{2}$ inches? Ans. $3{\cdot}5^3 \div 7\frac{1}{9} = 6{\cdot}0293$ lb.

2. What is the diameter of an iron ball which weighs 24 lb.?

Ans. $\sqrt[3]{24 \times 7\frac{1}{9}} = \sqrt[3]{170{\cdot}\dot{6}} = 5{\cdot}547$ inches the diameter.

3. What is the weight of an iron ball, of which the diameter is 4·6 inches? Ans. 13·688 lb.

4. What is the diameter of an iron ball which weighs 36 lb.?

Ans. 6·349 inches.

5. What is the weight of an iron ball, of which the diameter is 5·5 inches? Ans. 23·3965 lb.

6. What is the diameter of an iron ball which weighs 48 lb.?

Ans. 6·988 inches.

Prob. II. Given the diameter of a leaden ball, to find its weight, and the converse.

Rule. Divide the cube of the diameter by $4\frac{1}{2}$: the quotient will be the weight in pounds.

Multiply the weight by $4\frac{1}{2}$: the cube root of the product will be the diameter in inches.

1. What is the weight of a leaden ball, of which the diameter is 4·25 inches? Ans. $4·25^3 \div 4\frac{1}{2} = 17·059$ lb.

2. What is the diameter of a leaden ball which weighs 36 lb.? Ans. 5·45 inches.

3. What is the weight of a leaden ball, of which the diameter is 4·6 inches? Ans. 21·63 lb.

4. What is the diameter of a leaden ball which weighs 48 lb.? Ans. 6 inches.

PROB. III. To find the weight of an iron shell.

RULE. Take the difference between the cubes of the external and internal diameters, and divide it by $7\frac{1}{9}$: the quotient will be the weight in pounds.

1. What is the weight of a 13-inch shell, the inner diameter being 9 inches? Ans. $(13^3 - 9^3) \div 7\frac{1}{9} = 206·4375$ lb.

2. What is the weight of a shell, of which the diameters are 11·1 and 8 inches? Ans. 120·323 lb.

3. What is the weight of a 16-inch shell, the inner diameter being $11\frac{1}{2}$ inches? Ans. 362·127 lb.

4. What is the weight of a shell whose diameters are 15·4 and 11·2 inches? Ans. 316·032 lb.

PROB. IV. To find how much powder will fill a case.

RULE. Find the content in inches, and divide it by 30: the quotient will be the weight in pounds.

1. How much powder will fill a cubical box, of which the side is 18 inches? Ans. $18^3 \div 30 = 194·4$ lb.

2. How much powder will be contained in a cylinder which is 1 foot in length, and the diameter of its base 4 inches? Ans. 5·02656 lb.

3. How much powder will a chest hold, which is 15 inches long, 13 inches broad, and 5 inches deep? Ans. 32·5 lb.

4. What is the side of a cubical box which will hold 12 lb. of powder? Ans. 7·113 inches.

5. What is the side of a cubical box which will hold 24·3 lb. of powder? Ans. 9 inches.

PROB. V. To find how much powder will fill a shell.

RULE. Divide the cube of the internal diameter in inches by 57·3: the quotient will be the weight in pounds.

Multiply the weight by 57·3: the cube root of the product will be the diameter.

1. How much powder will a shell of 9 inches internal diameter hold? Ans. 57·3)729·0(12·7225 lb.

2. Required the diameter of a shell which will hold 9 lb. of powder. Ans. 8·02 inches.

3. How much powder will fill a shell, of which the inner diameter is $11\frac{1}{2}$ inches? Ans. 26·54 lb.

4. Required the diameter of a shell which will hold 15 lb. of powder. Ans. 9·51 inches.

PILING OF BALLS.

Balls and Shells are piled up in horizontal courses, upon a base of the form of an equilateral triangle, or of a square, or of a rectangle. The number of balls in a row diminishes, till, in the two first forms, it ends in a single ball, and in the last in a single row. The number of rows is equal to the number of balls in the lesser side of the under row. The number in the top row of a rectangular pile is one more than the difference between the length and breadth of the bottom row.

Prob. I. To find the number of balls in a triangular pile.

Rule. Multiply the number of balls in a side of the bottom row by that number increased by 1, and again by that number increased by 2: the product, divided by 6, will be the number of balls in the pile.

1. Required the number of balls in a triangular pile, of which each side of the base contains 30 balls. Ans. 4960 balls.

2. Required the number of balls in a triangular pile, each side of the base containing 64 balls. Ans. 45760 balls.

3. Required the number of balls in a triangular pile, each side of the base containing 80 balls. Ans. 88560 balls.

Prob. II. To find the number of balls in a square pile.

Rule. To twice the number of balls in a side of the bottom add 1, and multiply the sum by the number in that row, and by that number increased by 1: the product, divided by 6, will give the number of balls in the pile.

1. Let the side of the bottom row of a square pile contain 20 balls. How many balls are in the pile? Ans. 2870 balls.

2. Let the side of the bottom row of a square pile contain 80 balls. How many balls are in the pile?

Ans. 173880 balls.

3. Let each side of the bottom row of a square pile contain 50 balls. How many balls are in the pile?

Ans. 42925 balls.

Prob. III. To find the number of balls in a rectangular pile.

Rule. From 3 times the number in the length of the bottom row, increased by 1, subtract the number in the breadth, and multiply the remainder by the breadth, and by the breadth increased by 1: the product, divided by 6, will give the number of balls in the pile.

1. Suppose the number of balls in the length of a rectangular pile to be 59, and in the breadth 20. What is the number in the pile? Ans. 11060 balls.

2. Suppose the length contains 80, and the breadth 60. How many balls are in the pile? Ans. 110410 balls.

3. Suppose the length contains 100, and the breadth 75. How many balls are in the pile? Ans. 214700 balls.

Prob. IV. To find the number of balls in an incomplete pile.

Rule. From the number of balls in the complete pile subtract the number in the pile that is wanting, both computed as before: the remainder is the number in the incomplete pile.

1. Required the number of balls in a rectangular pile of 15 courses, the numbers in the bottom row being 60 and 25. Ans. 14590 balls.

2. Required the number of balls in a triangular pile of 15 courses, when each side of the base contains 60. Ans. 11605 balls.

3. Required the number of balls in a square pile of 20 courses, each side of the base containing 160. Ans. 453670 balls.

THE WORKS OF ARTIFICERS.

Artificers take the dimensions of their work with a measuring-line, divided into feet and inches, or by the carpenter's rule, or by a yard divided into inches and parts.

The work is generally computed by duodecimal multiplication, in which the inch is supposed to be divided into 12 parts, and each part into 12 seconds, &c.

Rule. Multiply each denomination of the multiplicand by the feet of the multiplier, and place the product under that denomination of the multiplicand from which it arises, carrying at 12. Then multiply by the inches of the multiplier, and set each product a denomination farther towards the right hand. Next multiply by the parts, if any, and set the products a place still farther to the right. Then add the products.

1. Multiply 9 f. 4 in. by 3 f. 8 in.

9	4		
3	8		
28	0		
6	2	8	
34	2	8	product.

2. Multiply	98	3		by	5	6		Ans.	540	4	6			
3. . . .	148	3		by	8	9			1297	2	3			
4. . . .	87	6	8	by	11	10		.	1036	0	10	8		
5. . . .	63	4	6	by	8	9	6	.	557	2	0	9		
6. . . .	55	8	7	by	72	6	3	.	4040	6	2	7	9	
7. . . .	105	3	4	by	27	9	6	.	2925	10	1	8		
8. . . .	208	7	9	by	12	5	4	.	2596	5	9	4		
9. . . .	365	11	8	by	13	6	3	.	4948	2	11	11		
10. . . .	185	10	9	by	15	9	8	.	2938	2	2	11		

Note. The feet in the product are square feet, 9 of which make a square yard, and 36 square yards make a rood of building. The inches in the product are 12th parts of a square foot, or each of them is 12 square inches, and the parts are square inches. The lower denominations are commonly expressed in fractions of a square inch: thus, 8 seconds are $\frac{2}{3}$ of a square inch, 9 seconds are $\frac{3}{4}$, and 7 seconds 6 thirds are $\frac{5}{8}$.

OF THE CARPENTER'S SLIDING-RULE.

The works of artificers, as well as the quantity of tim are often computed by the sliding-rule.

This rule consists of two pieces, each a foot long, faste together with a folding joint, with a slider in one of t pieces.

The edge of each piece of the rule is divided into 10 eq parts, and each part is subdivided into 10 equal parts; so t by it the dimensions may be taken in feet and decimals.

One of the faces is divided into inches, and 8th or 16 parts; and on the same face are several plane and diag scales, the diagonal being divided into 12 parts.

On the other face, the piece which has the slider conta four lines, two on the slider marked B and C, and two on t rule; one under the slider marked A, and the other abov marked D. The three lines A, B, and C, are of the s length, and divided in the same way: the divisions on D double of those on the other lines. These divisions are logarithmical; that is, if the distance between the first 1 the other 1 be divided into 1000 equal parts, the dist between 1 and 2 is 301 parts, which is the logarithm of 2, the distance between 1 and 3 is 477, the logarithm of 3, &c.

The first 1 may be read 1, or 10, or 100, and all the re are valued according to it. If it be read 1, the second 1 is 10, and the third 1 is 100, and then the first 2 is read 2, the second 2 is 20; but if the first 1 be called 10, the secon 1 is 100, and the third 1000, and then the first 2 is 20, the second 2 is 200. And all the other divisions and subdi visions are valued in the same way.

On the same face of the rule, there is on the other piece of it a table of the value of a load, or of 50 cubic feet of timber, at all prices, from 6d. to 2s. each foot.

PROB. I. To multiply numbers by the rule.

Set 1 on B opposite to the multiplier on A; then opposite to the multiplicand on B will be the product on A.

1.	Multiply 16 by 6.	.	Ans. 96.
2.	. . . . 23 by 14.	. . . .	322.
3.	. . . . 27 by 23.	. . . .	621.
4.	. . . . 68 by 46.	. .	3128.

PROB. II. To divide numbers.

Set the divisor on B to 1 on A; then against the divid on B will be found the quotient on A.

1. Divide 96 by 24.	. . .	Ans. 4.
2. . . . 576 by 48.	. . . .	12.
3. . . . 156 by 23.	. . .	6·8.
4. . . . 988 by 76.	. . . .	13.

PROB. III. To work a proportion.

Set the first term on B to the second on A; then against the third on B will stand the fourth on A.

1. Required the fourth proportional to 12, 28, and 114. Ans. 266.

2. Required the third proportional to 18 and 54. Ans. 162.

3. If 14 men build 4 roods, how many will in the same time build 28 roods? Ans. 98 men.

4. If 42 men perform a piece of work in 108 days, in what time will 72 do it? Ans. 63 days.

NOTE. This, with the two preceding rules, depends upon this principle: In a proportion, the difference between the logarithms of the first and second terms is equal to the difference of the logarithms of the third and fourth; and 1 is to the multiplier or divisor, as the multiplicand or quotient is to the product or dividend.

PROB. IV. To extract the square root.

Set 1 on C to 1 on D; then against the given number on C is its square root on D.

NOTE. The 1 on C must be read 1, or 100, or 10,000; and the 1 on D must be read 1, or 10, or 100, accordingly.

1. Required the square root of 576.	.	Ans. 24.
2. . . . . of 196.	. .	14.
3. . . . . of 4096.	. .	64.
4. . . . . of 9216.	. .	96.

PROB. V. To find a mean proportional between two numbers.

Set the less on C to the same number on D; then against the greater number on C will stand the mean proportional on D.

1. Required the mean proportional between 4 and 36.	Ans. 12.
2. . . . . 144 and 576.	288.
3. . . . . 513 and 57.	171.
4. . . . . 128 and 32.	64.

TO MEASURE TIMBER.

Prob. I. To find the area of a board.

Rule. Multiply the length by the mean breadth.

Note. When the board tapers regularly, half the sum of the breadths at the ends is the mean breadth.

By the Sliding-Rule.

Set the breadth in inches on B to 12 on A; then against the length in feet on B will be the content on A, in square feet and decimals.

1. Required the content of a board 12 feet 6 inches long, and 1 foot 3 inches broad. Ans. 15 feet 7 inches 6 parts.
2. Required the content of a board 13 feet 4 inches long, and 1 foot 8 inches broad. Ans. 22 feet 2 inches 8 parts.
3. Required the content of a board 11 feet 10 inches long, and 11 inches broad. Ans. 10 feet 10 inches 2 parts.
4. Required the content of a board 16 feet 9 inches long, and 2 feet 2 inches broad. Ans. 36 feet 3 inches 6 parts.
5. Required the content of a board 14 feet 11 inches long, and 9 inches broad. Ans. 11 feet 2 inches 3 parts.
6. Required the content of a board 10 feet 10 inches long, and 8 inches broad. Ans. 7 feet 2 inches 8 parts.

Prob. II. To find the content of squared timber.

Rule. Multiply the mean breadth by the mean thickness: the product, multiplied by the length, will give the content.

By the Sliding-Rule.

Find a mean proportional between the breadth and thickness. Then set the length on C to 12 on D; and against the mean proportional on D in inches will be the solid content in feet on C.

Note. If the quarter-girt be in feet, use 1 instead of 12 on D.

1. Required the content of a log, the length 24 feet 6 inches, mean breadth 1 foot 1 inch, and mean thickness 1 foot 1 inch. Ans. 28 feet 9 inches $\frac{1}{2}$ part.
2. Required the content of a log, the length 27 feet, mean breadth 1 foot 10 inches, and mean thickness 1 foot 3 inches. Ans. 61 feet 10 inches 6 parts.
3. Required the content of a log, the length 18 feet 6 inches, mean breadth 1 foot $4\frac{1}{2}$ inches, and mean thickness 1 foot 2 inches. Ans. 29 feet 8 inches $1\frac{1}{2}$ part.
4. Required the content of a log, the length 20 feet 6 inches,

mean breadth 1 foot $2\frac{1}{2}$ inches, and mean thickness 1 foot $2\frac{1}{2}$ inches. Ans. 29 feet 11 inches $2\frac{1}{8}$ parts.

5. Required the content of a log, the length 30 feet 8 inches, mean breadth 2 feet 1 inch, and mean thickness 2 feet 2 inches. Ans. 138 feet 5 inches $1\frac{1}{3}$ parts.

6. Required the content of a log, the length 40 feet 7 inches, mean breadth 2 feet 3 inches, and mean thickness 1 foot 9 inches. Ans. 159 feet 9 inches $6\frac{3}{4}$ parts.

PROB. III. To find the content of round timber.

COMMON RULE. Take one-fourth of the mean girt, and square it, and multiply it by the length for the content.

By the Sliding-Rule.

Set the length in feet on C to 12 on D; then against the quarter-girt in inches on D will be the content in feet on C.

NOTE 1. Tapering timber should be divided into pieces of eight or ten feet long, and these parts should be computed separately and added.

NOTE 2. In order to reduce the tree to such a circumference as it would have without its bark, a deduction is generally made of $\frac{1}{2}$ or $\frac{3}{4}$ of an inch for every foot of quarter-girt for young oak, ash, beech, &c.; but 1, or even $1\frac{1}{2}$ inch, must be allowed for old oak, for every foot of quarter-girt.

NOTE 3. The common rule gives the content too small, by 3 feet on every 11 feet of content; yet it is universally used in practice, being originally introduced in order to compensate the purchaser of round timber for the waste occasioned by squaring it.

RULE II. Take one-fifth of the girt, and square it, and multiply by twice the length for the content.

By the Sliding-Rule.

Set twice the length on C to 12 on D; then against $\frac{1}{5}$ of the girt on D will be the content in feet on C.

1. Required the content of a piece of round timber $9\frac{1}{2}$ feet long, and its mean girt 14 feet.

Ans. 116 feet $4\frac{1}{2}$ inches by the common rule; or, adding $\frac{3}{11}$ of it, the real content will be 148 feet 1·364 inch.

2. Required the content of a tree 24 feet long, and its girts at the ends 14 and 2 feet.

Ans. 96 feet by the common rule; the true content is 122·88 feet.

3. How much timber in a tree 18 feet long, and its mean girt 5 feet 8 inches?

Ans. Common rule 36 feet $1\frac{1}{2}$ inch; true content 46 feet 2 inches 10·56 parts.

4. How much timber in a tree 32 feet long, and its girts in the middle of every 8 feet are 64, 56, 52, and 46 inches?

Ans. 41 feet $10\frac{1}{6}$ inches by the common rule; true content 53 feet 6 inches 9·28 parts.

5. Required the content of a tree 30 feet long, the girts in the middle of every 10 feet being 50·4, 54·8, and 60·8 inches.

Ans. 40 feet 1 inch 2·9 parts by the common rule; true content 51 feet 3 inches 11·87 parts.

6. Required the content of a tree 55 feet long, the girts in the middle of every 11 feet being 72, 56, 42, 35, and 25 inches.

Ans. 56 feet 11 inches $8\frac{5}{8}$ parts by the common rule; true content 72 feet 11 inches 1·92 parts.

7. Required the content of a tree 50 feet long, its mean girt being 7 feet.

Ans. 153 feet $1\frac{1}{2}$ inch by the common rule; true content 196 feet.

8. Required the content of a tree 48 feet long, the girts at its ends being 60 and 18 inches.

Ans. 31 feet $8\frac{1}{4}$ inches by the common rule; true content 40 feet 6·72 inches.

9. Required the content of a tree 45 feet long, the mean girt being 74 inches.

Ans. 106 feet $11\frac{7}{16}$ inches by the common rule; true content 136 feet $10\frac{4}{5}$ inches.

10. Required the content of a tree $17\frac{1}{4}$ feet long, the girts in five different places being 9·43, 7·92, 6·15, 4·74, and 3·16 feet.

Ans. 42·5195 feet by the common rule; true content 54·425 feet.

MASON WORK.*

RUBLE WORK is measured in three different ways.

I. When the tradesman furnishes all materials.

Find the depth of the foundation at several places, and take the mean height from the foundation to the top of the side walls. Take the length of the side walls on the outside, and the breadth of the gables or cross walls on the inside of the building.

* The rules for the Mensuration of Artificers' Works, with the various allowances, have been furnished by an eminent surveyor in Edinburgh, and cannot fail to be of great advantage to the students for whom this section is intended. The allowances apply principally to Scotland; but the rules for taking the dimensions are applicable both to England and Ireland.

Gable-ends are measured by multiplying the height from the level of the side walls to the bottom of the chimney-stalk by half the sum of the breadths at the top of the side walls and at the bottom of the chimney-stalk; and the chimney-stalk is measured by multiplying half the girt by the height from the bottom of the stalk to the top of the cope.

Vents are measured by the lineal foot, from the top of the stalk to the bottom of the jambs.

Stormonts on side walls are measured by adding the thickness of one haunch to the length of the square part, and multiplying it by the height from the level of the side walls to the bottom of the angle; and the angular part and stalk are measured the same way as a gable-end and chimney-stalk.

All breaks and projections, whether external or internal, are found by adding one return to the length, and multiplying the sum by the height and thickness, and reducing it to the standard of the wall.

An allowance of 1 foot by 9 inches, multiplied by the length, is made for every levelling for joists and belts in ruble walls; and 1 foot by 2 feet, multiplied by the length, is made for levelling the tops of side walls, skews, and chimney-stalks; but no allowance is made for belts on ashlar fronts. An allowance of 9 inches square by the length is made for levelling for bond-timbers and ragulates for roofs in the chimney-heads only; 1 foot by 9 inches is allowed for ragulates left for stairs; and 1 foot by 6 inches for thin walls. These allowances must all be reduced to the standard of the walls in which they are made, and rated as workmanship only.

The daylight of all vacancies is to be deducted.

Rough stones more than 3 feet in length, placed as safes over voids, are to be taken by number, according to their different lengths.

Arches over cellars, &c. are taken by the girt of the soffit and the deepness of the arch-stones, once added for the breadth, and then by the length and thickness of the arch, and are double measure; and arches having been included in the general dimensions are to be again taken by their height, thickness, and length, and reduced to the standard of the wall.

All upright circular walls are double measure; and walls circular on one side only are allowed 1 foot thick round the circular part as double measure, and reduced to the standard of the wall, besides the solid content of the straight part.

The ruble of stair-steps and platts is taken by their length without the wall, and by their breadth and thickness, and in all cases reduced to 1 foot thick.

Ruble is allowed for all pavement, whether laid on lime or sand; and in no case is the thickness reckoned less than 4 inches.

In measuring separated pillars, when the face or front of the pillar does not exceed 5 feet in length, they are taken by their net height and length, and an allowance of 2 feet square by the height is made for carrying up the scontion. But this allowance applies only to pillars at and above 2 feet thick: all below that have the net thickness added to the length.

II. When the tradesman furnishes workmanship only.

The dimensions are taken over both side walls and gables, and no deduction is made for voids.

III. When the tradesman furnishes workmanship, lime, and sand.

The outside walls are measured by including the thickness of one side wall, and one-half of the vacancies is deducted.

Note. Ruble walls, in all the three cases, at and below 18 inches thick, are to be reduced to 1 foot, and all above 18 inches reduced to 2 feet thick, and measured by the rood of 36 square yards.

On doors and windows where there is no hewn work, an allowance is made of 1 foot square by the length, in name of hammer-dressed or cloured scontions.

HEWN WORK.

Hewn Work in Ruble Walls. The rybats of doors and windows are measured by girting from the bottom of the check outward, including the backset, if any. Soles and lintels are taken for the length over the face of the rybats, including the projection of one end, if projected; and the girt is taken as in the rybats.

Hewn corners are taken by the height for the length, and by the mean girt for the breadth.

Skews are taken by the length and by the girt, and chimney-copes by the extreme length all round, and for the breadth by girting from the open of the vent down to the chimney-stalk.

When the whole front of a building is of hewn or polished work, it is taken by the extreme length and height of the different species of work, including the sides of breaks, if any; but no allowance is made for the internal corners of such breaks. All voids are deducted; but the breasts and checks of rybats, together with the under bed and checks of the lintel, and upper bed of the sole, including their rests, are measured and added.

When architrave rybats are placed in a hewn front, the deductions are taken over these; and such moulded architrave rybats are measured by the height, and by girting from the bottom of the check outwards to the face of the plain ashlar. The lintels are girted in the same manner, and the length is taken round the ends.

Moulded architrave rybats of main doors, or otherwise, are taken in the same manner, and the whole reported as moulded work, excepting when plain ashlar stones are placed in the scontions, between the outband rybats and the checks; in which case these must be deducted, and added to the plain ashlar.

The Hewn Work of Arches is measured by finding the mean height of the arch stones, and for the length by laying the line round the middle of the face of the arch. The soffit and check are taken for the length round the check, and for the breadth by girting from the bottom of the check outward to the face of the arch. Both face and soffit are reckoned double measure. Arches in upright circular walls are allowed three measures.

When pannels are sunk on ashlar work, after they are included in the surface, the sunk part, and that round the edges, are taken over again; and if a moulding is round it, the whole is taken as moulded work.

All hewn work cut circular for skews is allowed 6 inches by the length for cutting.

Rustic work, whether square or champhered, is first measured superficially, and the checks or champhers are measured over again. Giblat checks, in like manner, are measured over again, after having been included in the face on scontions.

Pilasters, when they are raised out of the solid stones, and built in courses along with the ashlar, are girted in along with the ashlar, and the sunk part and edges are taken over again. If the pilasters are fluted, they are measured over again as moulded work, girting into the flutes and over the fillets. The cabled part, if any, is measured in the same way, and allowed double measure. The bases and capitals are girted as mouldings.

Columns, of which the shafts are diminished with a curve or swell, are allowed double measure and a half; and if the neck-moulding is wrought on the shaft, they are allowed three measures. When the shafts are diminished straight, without a swell, double measure is only allowed, and a half more if the neck-moulding is wrought on the shaft. The fluted and cabled parts of columns are measured the same as in pilasters, after they are taken for plain work, as above. The bases and

capitals are girted as other mouldings, with the usual allowance; and the number and size of carved capitals must be given.

Cornices are taken for their length at the extremities of their greatest projection, and for their breadth by girting their mouldings; and so much of the superficies of the upper bed as is without the wall is added.

Block and dental cornices, after being measured in the same manner, have the backs and soffits of the recesses, together with the ends of the blocks and dentals, added.

The steps of hanging stairs, whether moulded or plain, are girted at their mean breadth, including both joints, and for their length what is seen, including 6 inches of rests in the wall. The soffit and ends of wheel steps are taken over again, so far as is without the walls; and the ends of both square and wheel steps are taken at their extreme breadth and depth.

The joints of platts, if joggled, are also taken.

The skifting of hanging stairs is taken by the extreme length and breadth of the stones, including the upper edge.

The steps and platts of newal stairs are taken by girting at the mean breadth, allowing 1 inch of overlap on each step, and for the length by what is seen, allowing 6 inches on each end for rests. The newals are girted round, including the backset. The tails are taken as scribbled work, and the soffits of steps according to the kind of work upon them.

Pyramids or obelisks, if they are built in courses, are girted for the length at the bottom of each course, and between the joints for the breadth, and allowed measure and half. When they are made of one stone, they are girted for the length at the bottom, and for the breadth by the sloping height; and if they are polygonal figures, the peends or champhers are taken over again.

In measuring cirb-stones, besides the upper bed, 6 inches are allowed on the edge, of the same work with the upper bed.

Hewn work of every kind, as well as coursed, cloured, or scribbled work, is measured by the superficial foot.

NOTE. In measuring harling, the whitewashing on the faces and breasts of rybats, belts, chimney-copes, &c. is taken as harling.

1. How much ruble work of the standard thickness of 2 feet is in a house of 3 stories, 60 feet long and 30 feet broad within walls, the height 30 feet from the foundation to the top of the side-walls, 12 more to the foot of the chimney-stalks, which are 7 feet high, 10 broad, and 3 feet thick, the skews are 14 feet 6 inches long, the side-walls $2\frac{1}{2}$ feet thick, and the end-walls 3 feet thick, with two doors, each 7 feet by 4 feet,

22 windows, each 5 feet by 3 feet, and 12 windows, each 4 feet by $2\frac{3}{4}$ feet?

A side-wall,	$66\times30\times1\frac{1}{4}$	$=2475$ sq. feet.
An end-wall,	$30\times30\times1\frac{1}{2}$	$=1350$
A gable-end,	$\frac{1}{2}(35+10)=22\frac{1}{2}\times12\times1\frac{1}{2}$	$=405$
A chimney-stalk,	$10+3=13\times7\times1\frac{1}{2}$	$=136\frac{1}{2}$
		$4366\frac{1}{2}$
		2
		8733
Levelling side-walls,	60×2	$=120$
Ditto for joists,	$60\times\frac{3}{4}\times2$	$=90$
Ditto 4 skews,	$14\frac{1}{2}\times4$	$=58$
Ditto chimney-tops,	10×2	$=20$
		9021

2 doors,	$7\times4\times2$	$=56$	
22 windows,	$5\times3\times22$	$=330$	
12 windows,	$2\frac{3}{4}\times4\times12$	$=132$	
		518	
Add $\frac{1}{4}$ for thickness,		$129\frac{1}{2}$	
			$647\frac{1}{2}$

Content 25 roods 30 yards $3\frac{1}{2}$ feet, $8373\frac{1}{2}$ sq. feet.

2. How much hewn work in 22 window-lintels and soles, each 4 feet by $1\frac{2}{3}$ feet; 12 lintels and soles, each $4\frac{1}{2}$ feet by $1\frac{1}{3}$ feet; two door-lintels and soles, each 5 feet by $1\frac{2}{3}$ feet; 22 pairs of rybats of 5 feet, 17 rybats of 4 feet, and 2 ditto of 7 feet, all of them 14 inches broad; skews 64 feet by 14 inches, coping of chimney-stalks 44 feet by 20 inches, and coping of the roof 60 feet by 16 inches?

Ans. $687\frac{1}{3}$ square feet.

3. What should be charged for the workmanship of a house of 2 stories, 36 feet long and 24 feet broad within the walls; the side-walls 2 feet thick and 24 feet high, measured from the foundation; the gables 3 feet thick and 16 feet higher than the side-walls to the bottom of the chimney-stalks, which are 7 feet broad, 3 deep, and 8 high; the skews are 19 feet 3 inches in length, and also 110 feet of vents: the ruble work, reduced to the standard, is at £2 per rood, and the vents at 4d. per foot. Ans. £34, 11s. $2\frac{3}{4}$d.

4. A house of 3 stories is 45 feet long and 28 feet broad within walls, and the height from the foundation to the top of the side-walls is 30 feet; the gables rise 18 feet above the

side-walls to the bottom of the chimney-stalks, which are 8 feet wide, 3 deep, and 10 high; the skews are 21 feet 11 inches long; the side-walls are $2\frac{1}{2}$ feet, and the gables are 3 feet thick; there are 2 doors in the sides, each $7\frac{1}{2}$ feet by 4 feet; 12 windows in the sides, and 6 in the ends, each 6 feet by $3\frac{1}{2}$ feet. Required the expense of the materials and workmanship of the ruble work, at £10, 6s. 8d. per rood, allowing £2, 14s. per rood for levelling the side-walls.

Ans. £231, 18s. $0\frac{1}{4}$d.

5. A house is 41 feet long, $20\frac{1}{2}$ feet broad within the walls, and 18 feet 9 inches high from the foundation to the top of the side-walls, which are 2 feet thick; the gables are $2\frac{1}{2}$ feet thick, and rise 8 feet 6 inches above the side-walls to the bottom of the chimney-stalks, which are 4 feet wide, $2\frac{1}{2}$ feet thick, and 5 feet 1 inch high. The broached hewn work consists of 4 skews, each 11 feet 6 inches by 1 foot 7 inches; 4 corners, each 18 feet 9 inches by $2\frac{1}{2}$ feet; and 2 chimney-stalks, the girt of each 13 feet, and the height 5 feet 3 inches. The droved hewn work consists of the rybats and lintels of 6 windows, each 13 feet 11 inches by 15 inches; 6 soles of ditto, each 3 feet 11 inches by 19 inches; the rybats and lintels of one window, 9 feet 3 inches by 15 inches; sole of ditto, $3\frac{1}{4}$ feet by 19 inches; the rybats and lintel of a door, $19\frac{1}{4}$ feet by 15 inches; sole of ditto, $4\frac{1}{4}$ feet by 19 inches; 3 pairs of jambs, each 6 feet by 2 feet; the lintels of ditto, each 4 feet 5 inches by 15 inches; 3 inner hearths, each 3 feet 1 inch by 18 inches; 3 outer hearths, each 3 feet 8 inches by 20 inches; kitchen jambs, 8 feet 8 inches by 2 feet 3 inches; lintel of ditto, 5 feet 8 inches by 15 inches; the hearth, 4 feet by 21 inches; and also $106\frac{1}{2}$ feet of vents. Required the content of the ruble work and of the hewn work, and also the expense of the workmanship; the ruble work being at £3 per rood, the broached hewn work at 5d. per foot, the droved hewn work at 6d. per foot, and the vents at 6d. per foot.

Ans. 10 roods 17 yards 8 feet $10\frac{1}{4}$ inches ruble work; 396 feet 10 inches broached hewn work; 307 feet $5\frac{1}{4}$ inches droved hewn work. Expense £50, 2s. $3\frac{1}{4}$d.

BRICK WORK.

Brick Work is measured by the square yard, and reported as brick on edge or brick on bed, 9 inches or 14 inches thick; and all above that is reduced to 14 inches as the standard.

Brick walls are measured the same way as stone walls, and the daylight of all vacancies deducted.

Upright circular walls and arches are allowed measure and

half; and arches over voids in upright walls are taken over again. Groin-arches are double measure; and 18 inches by the length and thickness allowed on the groin for cutting.

The tops of niches and spherical arches, whether of brick or stone, are allowed three measures.

When the skews on brick gables are feathered on edge or feathered and tongued, they must be taken over again; and in all cases, $4\frac{1}{2}$ inches by the thickness of the skew above the thackgate is allowed for cutting. Chimney-stalks are taken by the height and by the breadth, adding in the thickness of one haunch, if it does not exceed 18 inches; and in all above that thickness one-half of the haunch is added.

1. The sides of a brick vault are 18 feet long and 5 feet high, and 2 bricks thick; the girt of the arch 10 feet, and 1 brick thick; the end walls 8 feet long, 7 feet high, and $1\frac{1}{2}$ brick thick; the door 5 feet by $2\frac{1}{2}$ feet. How much does the vault measure at standard thickness?

Ans. 63·3183 square yards.

2. How many square yards of standard brick work in a wall 75 feet long, 15 feet 9 inches high, and 3 bricks thick?

Ans. 262 yards $4\frac{1}{2}$ feet.

3. A garden is 160 feet broad, and contains an acre. Required the expense of enclosing it with a brick wall 10 feet 6 inches high, and $2\frac{1}{2}$ bricks thick, at 5s. $7\frac{1}{2}$d. per square yard of standard thickness, deducting 2 doors, each 6 feet 9 inches by 4 feet, and a gateway 11 feet wide. Ans. £463, 18s. $10\frac{3}{4}$d$\frac{1}{2}$.

CARPENTERS' AND JOINERS' WORK.

Common rough joisting is measured by adding to what is in sight the rests in the walls; and when that cannot be ascertained, an allowance not exceeding one foot on each end is made, and the content is estimated in square yards, stating the size and distance.

Framed joisting is measured in the same way for the scantling or bridged joists, and the surface-measure includes the beams. But beams and transoms are measured by the cubic foot. When joists are laid on the tops of walls, and the ends of couples joined to them, or when beam-fitted, and the wall-plates fixed down to them,—in both cases they are taken as joisting.

Trussed and dressed beams are measured by the cubic foot, and the oak in trussed beams is reported lineal, stating the size. Dwangs put between joists are classed with rough timber, such as safe-lintels, &c.

Deafening-boards are measured superficially; and when deafening is laid, the hearth-places are deducted, and the boxing for the hearths stands as an equivalent for the floor.

Flooring is measured superficially, and reported according to its quality, as deal, or batten-flooring, &c. No deduction is made for hearths where there is strong boxing under them not measured separately; but when that is the case, the hearths are deducted. When floors are cut to any angle or circle at or exceeding 45°, an allowance of 6 inches by the length is made for cutting. Hearth-borders are taken by the lineal foot.

Framed and bound roofing is measured by taking all the principal timbers by the cubic foot that are connected with the main couples, and also the extra size of diagonals, when they are above 9 inches by 3 inches, and reported as cubic framed timber.

The surface of a square roof is measured by taking twice the depth from ridge to eave, and by the length from skew to skew.

A pavilion or hipped roof is taken by adding the length and breadth at the eaves to the length at the ridge, or to the length and breadth of the platform, and by the depth from ridge to eave. The platform is taken as flooring and joisting. A pavilion square roof, finishing in a point at the top, is taken by girting one end and one side at the eave for the length, and the depth, as before.

A conical or turnpike roof is measured by taking the circumference at the eaves, and by the slant height.

Eighteen inches by the length are allowed for each peend and flank. All openings for stormont-windows, skylights, and chimney-stalks, are deducted, except when the opening is at or under 2 feet square; and when such deductions are made, there are 9 inches for the width of it allowed at top and bottom, for bridling.

The contents of the wall-plates, including the belgates built in for fixing them, are added to the surface-measure of the roof, unless when the wall-plates are above 2 inches thick; in which case they are taken as rough cubic timber.

The putting on of the iron-work of framed roofs ought to be included in the price; and, if furnished by the tradesman, charged by weight.

When there are two baulks in a common roof, the upper ones are included, and the under ones taken as joists, mentioning their size and distance.

Roofing and tile-lath are also measured superficially; and when sarking is put on slate eaves, it is measured as sarking

only. Roofs upon circular walls are allowed double measure, and all domes three measures.

Roofs put upon polygons, when the scantlings are curved, are allowed double measure.

Battens for ridges and peends, whether square or rounded, and filleting for skews, are measured by the lineal foot, specifying the size.

Framing for brick partitions, if the standards are placed at regular distances, is measured superficially by the yard, as brick on edge, brick on bed, stating the distance of the standards; and when dressed door-standards are placed in such partitions, the voids are deducted over the dressed standards.

When there are only a few detached standards in partitions, these are calculated to 3 inches square, and reported as rough standards; and the warpings, in that case, are to be reduced to 4 inches broad, and their thickness stated.

Standards for lath partitions are measured by the square yard, stating their size and distance, and deducting the doors over the door-standards, where dressed ones are placed. Runtrees at top and bottom (if any) must be reduced to 3 inches square, or 4 inches broad, stating the thickness.

Wall-battens for lath are measured by the superficial yard, and, if fixed on dooks, are reported as such.

All standards set circular are allowed measure and half. Bond-timbers are taken by the lineal foot, stating the general size.

Lathing is measured by the square yard, and, when on circular walls, allowed measure and half. All arches and coves are allowed the same.

Domes and tops of niches are double measure, unless otherwise specified. An allowance of 6 inches by the length is given for cutting round circles and angles; and all vacancies are deducted.

Dressed door-standards in brick or lath partitions are measured by their actual height, and, along with the lintels, reduced to 3 inches square.

All dressed posts or standards, at or below 6 inches square, are to be reduced to 3 inches square; from 36 inches square, to be reduced to 6 inches square; and all above that to be cubical.

Dressed deal door-breasts or hingings, not exceeding 8 inches broad and $1\frac{1}{4}$ inches thick, are reduced to 4 inches broad, and reported according to the thickness. All above 8 inches broad are reported by the superficial foot, as an article by itself; and all above $1\frac{1}{4}$ inches thick are reduced to 3 inches square.

Grounds are measured by the lineal foot, specifying whether thick or thin, or if checked or grooved.

Sash-windows are allowed 2 inches more than the daylight for the height, and 3 inches for each side-facing more than the daylight for the breadth, when they are not more than 3 feet wide: all above that are allowed 1 inch on each side of the facing for every foot in width.

Windows with circular tops are allowed double measure for the circular part. Convex or concave windows are double measure; and, if made to fit an arch on the top, the arched part is taken at its extreme height and breadth, and allowed three measures. Flat segment-topped windows are allowed 9 inches for cutting; and, when the peends are square, are taken as windows without glass. Cupola lights with curved ribs or astragals are allowed three measures; but when straight, only two. Common skylight hatch-windows by the surface.

The sash part of doors is measured by adding as much of the belt-rail to the height as the breadth of the stiles, and the remaining part is taken as bound work.

Chinese sash-lights are allowed double measure when the panes are of various figures, or circular; and if in a circular door, three measures are allowed; but only single measure when the panes are all one figure and one size.

Bound doors are measured by adding as many inches to the height as there are pannels in the height, and by the net breadth; and when the thickness is at or above $1\frac{3}{4}$ inches, double measure is allowed; below that thickness, when dressed on both sides, measure and half; but when dressed only on one side, no more than single measure.

Bound window-shutters are measured in the same way: if cut, two thicknesses are added to the length; and if checked for backfolds, the girt of one checked edge is added to the net breadth of both shutters.

Bound flush-and-bead shutters are measured by the square foot, specifying the thickness. Plain deal backfolds have the breadth of the cross heads added to the height, and are reported by the yard; and, if not more than 6 inches broad, they are reduced to 4 inches broad.

All circular bound work is allowed double measure.

Bound flush-and-bead doors, having two leaves, are measured like shutters with backfolds; but in shop-doors the sash is deducted from the bead-and-flush, and the sash and shutters taken by themselves.

Torus mouldings on bound work are taken by the lineal foot.

Plain deal-backed work, or double deal doors, &c., is taken

by the square foot; and if beaded on the joints, it should be specified, as well as the thickness.

Common plain deal, if dressed on both sides, is allowed measure and half, and reported by the square yard, stating the thickness; and whenever beads are put on the joints, half an inch is allowed in the measure for each bead.

Bound dado-lining is allowed on the length one inch for every external corner, for nailing, and an inch of cover for every architrave; and on the height, besides an inch for the pannels, 3 inches more than the net measure between the base and surbase, and no notice taken of the stile-ends.

Plain dado and window linings, when done in a superior manner, are reported as such by the yard, stating the thickness, and the bars behind are included in the price.

Shelving in general is taken by the yard, stating the thickness. When cut circular, the net area is taken, and an allowance of 3 inches on each edge for cutting; and when circular on one edge, an allowance of 6 inches is made for cutting. When shelves are wrought on both edges, they are allowed measure and half; and grooves for shelves are reported by the lineal foot.

Plain deal work, dove-tailed, is measured by the yard, stating the thickness and quality; and all broad plain deal work, of whatever description, above $1\frac{1}{2}$ inches thick, is taken by the square foot, and the thickness stated.

Mouldings are taken by their greatest length, and for their breadth by girting over the mouldings, allowing an inch more than is seen on base-mouldings for a rest on the plinth, and another allowance of one foot for every mitre more than four on base and surbase in one room.

The blocks on which architraves are set are included in with the height of the architraves, and then taken over again as skirting, along with the base-plinth.

Cornices of doors and chimney-pieces are taken at their greatest projection for the length, and by girting the moulding for the breadth; the upper bed being taken as moulded work as far back as the projection, the remaining part to be of plain deal, if there be any.

The frize-board is taken as plain deal, by the square foot, including what is behind the cornice; but when the frize is under 6 inches broad, with an astragal at the bottom, the whole is taken as mouldings.

All mouldings, except small single ones, are estimated by the superficial foot; dentills, Doric bells, &c. by the lineal foot.

The shafts of plain pilasters are taken by their extreme

height and breadth, and estimated by the square foot, stating the thickness, and both edges are girted on the face. Fluted pilasters are taken in the same way, girting over the fillets and into the flutes; and if the edges or returns are fluted, they are also girted in; but if they are planted returns, not fluted, they are taken as plain work, when above 2 inches broad. Cabled or reeded pilasters are taken as such, and the thickness in all cases stated. The bases and capitals of both plain and fluted pilasters are taken by themselves, as the mouldings of the pilasters.

Solid columns are taken by their height and greatest diameter; and when their mouldings are turned out of the solid, the diameter is taken at the base. The shafts of built columns are taken superficially by the whole height, and by the girt of the greatest diameter, and allowed two measures. When columns are fluted and reeded, they are taken as such; and if reeds are planted in, they are taken lineally. The bases and capitals are measured as mouldings, and allowed double measure.

Facings, skirtings, base-plinths, and door-stops, under 8 inches broad, are reduced to 4 inches broad; and all above 8 inches broad are taken as plain linings.

The stanchel part of railing is taken by the yard, stating the size of the stanchels and distance between; the posts and rails are reduced to 4 inches broad. The posts of the rail are included in the surface-measure.

The Chinese part of railing is measured by the square yard, as such, and the posts reduced to 3 inches square, and the rails to 4 inches broad, stating the thickness.

The square steps of timber stairs are taken by their length and by girting over the step and breast, allowing an inch of cover to each. The wheel steps are taken at their extreme length, and by girting at the mean breadth, allowing 3 inches on each step for cutting. Spring-boards and brackets are taken by the square foot, specifying their thickness.

Stair hand-rails are taken by the lineal foot, stating the quality. Circular parts are double measure, twist and circle three measures, and the measure taken round the scroll.

NOTE. In measuring rough cubical timber, one inch is allowed on the whole girt for bark, and no rough timber under 6 inches diameter is accounted measurable.

1. What is the value of a sash-window which measures 6 feet 10 inches by 3 feet 8 inches, at 2s. per square foot?

Ans. £2, 10s. $1\frac{1}{3}$d.

2. How many square yards of roofing and sarking are in a house 60 feet long from skew to skew, and each side of the

roof 22 feet, allowing 9 inches for the breadth of the wall-plate; and what is the value of it, at 9s. 6d. per square yard?

Two sides	45 feet	6 inches.
Length	60	0

9)2730(303$\frac{1}{3}$ square yards at 9s. 6d. = £144, 1s. 8d.

3. How many yards of flooring in a house of three stories, 56 feet by 28 feet within the walls, deducting the vacancy for the stair, 13 feet by 8 feet; and what is the value, at 5s. 6d. per square yard? Ans. £134, 4s.

4. How much wainscoting in a room 25 feet by 18 feet, and 14 feet 3 inches high when girt over the mouldings, allowing a door 7 feet 2 inches by 3 feet 4 inches, and 2 windows with shutters, each 5 feet 8 inches by 3 feet 6 inches, and a chimney 6 feet 4 inches by 5 feet 6 inches; the doors and shutters being charged work and half work?

Ans. 135 yards 7$\frac{1}{2}$ feet.

5. A partition is 173 feet 10 inches in length, and 10 feet 7 inches in height. How many squares are in it?

Ans. 18·39736 squares.

6. How many yards of flooring and joisting in a house of 3 floors, 48 feet by 27 within walls, allowing 9 inches for the rests of the joists, and deducting from each floor the vacancy for the stair, 12 feet by 8 feet 3 inches; and what is the expense of the materials and workmanship, the joisting and flooring at 7s. 6d. per yard, and the naked joisting at 3s. 6d. per yard? Ans. £153, 16s. 6d.

PLASTER WORK.

Plain plaster work is measured by the square yard, stating the number of coats and the quality of the finishings.

Upright circular walls, soffits of arches, coves, &c. are allowed double measure. Domes and tops of niches are allowed three measures. When new and old plaster are joined, an allowance is made of one foot for splicing; and when mouldings are put on plain plaster, to form pannels, the whole wall is taken as plain plaster, and the mouldings are taken again by the lineal foot.

Where stiles are raised, the general superficies of the wall is measured as pannelled plaster. The stiles and mouldings are taken by the lineal foot, stating the breadth.

These rules apply to ceilings as well as walls, and to mouldings, whether plain or enriched.

All circular mouldings on domes are double measure.

Pannelled soffits of arches, and pannelled scontions of stair-windows, are taken by girting over the mouldings both ways; and if at or above 12 inches broad, they are estimated by the square foot; but if under 12 inches, by the lineal foot, stating the breadth.

Architraves of arches are taken as other mouldings.

Plain cornices, at or above 12 inches in girt, are taken by the square foot, and all under that by the lineal foot.

Enriched cornices are measured in the same way, stating the number and nature of the enrichments; and for all mitres in a room, &c. more than four, one foot is allowed for each, whether external or internal.

Plain and enriched entablatures are measured by the square foot, by girting from the ceiling down to the plain plaster of the walls; and the number and quality of the enrichments must be stated.

Entablatures on the bottom of coves are measured on the upper bed, as far as the mould goes back, and down to the plain plaster.

If the ornaments and mouldings on a ceiling do not exceed 12 inches in their distance from each other, the whole ceiling is taken by the superficial foot, as an ornamented one; but when their distance exceeds 12 inches, the mouldings and margins are taken in the same way as pannelled plaster.

Centre ornaments above 3 feet diameter are taken by the square foot, and all at or under that by the piece, stating the size.

Heads, trusses, and other detached ornaments, are reported by the number and size.

Plaster beads are taken as plain mouldings, and relieved corner beads by the lineal foot, as double cut.

1. How much plastering on a partition 7 feet 8 inches long and 10 feet 3 inches high, deducting a door 6 feet 3 inches by 2 feet 10 inches; and what will it cost, at 5d. per yard?

	10 feet	3 inches.	6 feet	3 inches.
	7	8	2	10
	78	7 wall.	17	$8\frac{1}{2}$ door.
	17	$8\frac{1}{2}$ door.		
9	60	$10\frac{1}{2}$ content.		

6 yards 6 feet $10\frac{1}{2}$ inches content, at 5d. is 2s. $9\frac{3}{4}$d.

2. How many square yards of plastering on the walls and

ceiling of a room 30 feet long, 25 broad, and 12 high, deducting 3 windows, each 8 feet 2 inches by 5 feet, 2 doors, each 7 feet by 3 feet 6 inches, and a fireplace 4 feet 6 inches by 4 feet 10 inches, the sides of the windows behind the shutters being plastered, and measuring 8 feet 2 inches by 15 inches; and what will it cost, at $6\frac{1}{4}$d. per square yard?

Ans. 215 yards 3 feet, cost £5, 12s. $1\frac{3}{4}$d$\frac{1}{3}$.

SLATERS' WORK.

SQUARE roofs are girted for their deepness from the top of the ridge downwards, allowing 9 inches for the double eaves, and for the length between the skews, and 6 inches more for cover.

Chimney-stalks, and all voids above 4 square feet of daylight, are deducted, but allowing the double eaves above such openings, and also 9 inches for cutting along each side; but no deductions are made at or under 4 square feet.

Stormont and roof windows are measured according to the form of the different parts, and 9 inches by the length allowed for every cutting on peends, flanks, and skews.

Close flanks made waterproof without lead are allowed double of a common flank for cutting.

Circular work and dome roofs are double measure. Ridge stones are reported by the lineal foot.

Tile roofs are measured in the same way as slate roofs, but no allowance for double eaves, unless when slate eaves are put on, in which case 6 inches more than what is seen is allowed on the slating for cover.

The pointing of slate or tile roofs is measured as before-stated, but no allowance for cutting or for eaves. The deepness of the plaster is to be added to the length of the roof.

Slate and tile roofs are estimated by the rood of 36 square yards.

1. How much slating is in a roof 46 feet long, and 18 feet from the coping to the eaves? Ans. 5 roods 11 yards 6 feet.

2. How much slating is on the roof of a square building with a platform, the length at the eaves 72 feet, and at the platform 40 feet; the breadth from the platform to the eaves 12 feet, and along the hips $14\frac{1}{2}$ feet?

Ans. 8 roods 34 yards $1\frac{1}{2}$ feet.

3. Required the content of a tile roof 42 feet 7 inches long, and 16 feet 10 inches from the ridge to the eaves; and what does it amount to, at £3, 15s. per rood?

Ans. £16, 11s. $10\frac{1}{4}$d.

4. Required the expense of a slate roof measuring 48 feet 6 inches in length, and 24 feet from ridge to eaves, breadth of the wall-plate 9 inches, reckoning the roofing and sarking at 7s. per square yard, and the slating, including slates, at £5, 8s. per rood. Ans. £133, 7s. 6d.

PAINTERS' WORK.

Plain painting is measured wherever the brush touches, and estimated by the square yard, stating the colour and quality, whether oil or size, and the number of coats.

Party-coloured work is measured first as plain work, and then the stiles and mouldings are taken and estimated by the lineal foot, according to the number of different colours; and this rule applies to skifting and mouldings of a room, when different colours form the general body of the work.

An allowance of 6 inches for each enrichment in cornices is added to the girt, when enriched cornices are picked in; and if at or above one foot of girt, they are taken by the superficial foot; all under that girt by the lineal foot. In both cases, the number of enrichments are to be stated, besides being included along with the plain work with which it may class.

Ornamented ceilings are measured in the same way as plaster work.

Mock mouldings in passages, staircases, &c. are reported by the lineal foot. Outsides of windows are allowed one-fourth more than the net daylight.

Stanchel-railing, at or under 6 inches in the open, is allowed double measure; above 6 and under 9 inches, measure and half; from 9 to 12 inches, one and one-fourth; and all above that, single measure. Stanchels put into windows are taken by including one of the side spaces between the stanchel and the rybats.

Ornamented railing on stairs is allowed double measure, and figures of every description are reported by number.

1. How much painting on a wall 14 feet by $9\frac{1}{2}$ feet, deducting the chimney, 4 feet 6 inches by 3 feet 10 inches; and what does it come to, at 10d. per square yard?

Ans. Content 12 yards $7\frac{3}{4}$ feet, value 10s. $8\frac{1}{2}$d.

2. A room is 20 feet long, 14 feet 6 inches broad, and 10 feet 4 inches high. How much painting is in it, deducting a fireplace 4 feet 4 inches by 4 feet, and 2 windows, each 6 feet by 3 feet 2 inches? Ans. 73 yards $0\frac{2}{3}$ foot.

3. Required the expense of painting a room 28 feet long

and 20 broad, the girt of the wainscoting or *dado-work* round the bottom of the room 2 feet 10 inches by 84 feet; the height from the wainscoting to the ceiling 7 feet 10 inches; 3 windows, each 7 feet 10 inches by 4 feet 9 inches; 2 doors, and 2 presses, each 7 feet 6 inches by 4 feet; and a fireplace 4 feet 9 inches by 5 feet. The wood work is painted in oil at 9d. per square yard, the window-shutters and doors on both sides; the walls with size at 3d., and the ceiling is white-washed at $1\frac{1}{2}$d. per yard. Ans. £4, 1s. $11\frac{3}{4}$d$\frac{2}{3}$.

GLAZIERS' WORK.

Glass is measured by the superficial foot, stating the quality. Every pane is measured at the extreme points, including the back-check of the astragal.

1. A window is 5 feet 4 inches by 3 feet 2 inches of day-light. What does the glazing amount to at 14d. per square foot? Ans. Content $16\frac{8}{9}$ feet, value 19s. $8\frac{1}{2}$d.

2. An oval window is 4 feet 3 inches by 2 feet 5 inches. Required the expense of glazing it, at 1s. 3d. per square foot.
Ans. Content $10\frac{15}{48}$ feet, value 12s. 10d.

3. Required the expense of glazing the windows of a house of three stories, at 1s. 4d. per square foot, the common breadth of the windows being 3 feet 10 inches, and the height of the lower tier 7 feet 8 inches, of the second 6 feet 10 inches, and of the highest 5 feet 3 inches; 4 windows in each tier.
Ans. £20, 3s. $9\frac{1}{3}$d.

PLUMBERS' WORK.

Plumbers' Work is generally done by the pound or hundredweight; but the laying down of lead is done by the day.

Sheet-lead used in roofing, &c. is from 7 to 12 lb. per square foot. Leaden pipes of $\frac{3}{4}$ inch bore weigh 10 lb.; of 1 inch bore, 12 lb.; of $1\frac{1}{4}$ inch bore, 16 lb.; of $1\frac{1}{2}$ inch bore, 18 lb.; of $1\frac{3}{4}$ inch bore, 21 lb.; and of 2 inches bore, 24 lb. per yard in length.

1. Required the expense of a leaden pipe of $1\frac{1}{4}$ inch bore, and 72 feet long, at $3\frac{1}{4}$d. per lb. Ans. £5, 4s.

2. Required the expense of lining a water-cistern 2 feet 10 inches long, 2 feet 6 inches deep, and 2 feet broad, with sheet-lead of 10 lb. to the square foot, at £1, 18s. 9d. per cwt.
Ans. £5, 3s. $2\frac{1}{2}$d$\frac{15}{28}$.

3. The platform on the roof of a square building measures 40 feet square, and is covered with lead of 9 lb. to the square

foot; the hips are each 16 feet 6 inches long, and covered to the breadth of 18 inches with lead of 10 lb. to the square foot; the water-pipe is of 1 inch bore and 48 feet long, and the soil-pipe is of 2 inches bore and 30 feet long; the water-cistern is 3 feet 6 inches long, 2 feet 6 inches deep, and 3 feet wide, and lined with lead of 11 lb. to the square foot. Required the expense of the whole, the sheet-lead being rated at £1, 11s. 6d. per cwt., and the pipes at $4\frac{3}{4}$d. per lb.

Ans. £231, 12s. $5\frac{1}{2}$d$\frac{1}{2}$.

PAVIERS' WORK.

Causewaying is measured by the rood or yard, stating whether ruble or coursed work. One foot by the length is added as an allowance for every channel, and 6 inches by the length for cutting on coursed work, and for warpings.

Hewn pavement is measured by the square foot, stating the quality; and, if grooved pavement, the grooves are added to the surface-measure.

The hollow part of gutters cut in pavement is taken over again; and sinks are taken two times, after being included in the surface-measure.

1. A court-yard is 50 feet long by 40 feet 6 inches broad. What will the paving of it amount to, at 3s. $7\frac{1}{2}$d. per square yard? Ans. £40, 15s. $7\frac{1}{2}$d.

2. What will be the expense of paving a square court, the length of the side being 150 feet? The outside, to the breadth of 10 feet, is paved with Arbroath pavement at 3s. per square yard, and the rest is done with common pavement at 1s. 9d. per yard. Ans. £257, 12s. $1\frac{1}{3}$d.

3. A hexagonal space, the outside of which, to the breadth of 12 feet, in a line from the corner to the centre, is to be paved with Arbroath pavement at 2s. $10\frac{1}{2}$d. per yard; the rest, deducting a circular garden in the centre, of 300 feet diameter, is to be done with common pavement at 1s. $8\frac{3}{4}$d. per yard. Required the amount of the expense, supposing the length of the side 250 feet. Ans. £977, 14s. 1d.

OF VAULTS.

Vaults are formed by arches springing from the opposite walls, and meeting in a line at the top.

Prob. I. To find the surface of a vault.

Make a line ply close to the arch, from one side to the other, to get the girt, and multiply it by the length of the

vault to get the surface; and this, multiplied by the thickness of the arch, will give the solid content of the arch.

1. Required the surface of a vault 106 feet long, and the girt of the arch $42\frac{2}{5}$ feet. Ans. $499{\cdot}3\dot{7}$ yards.

2. Required the surface of a vault 56 feet long, and the girt of the arch 36 feet 4 inches; and also the solidity of the arch, its thickness being 3 feet.

Ans. $226\frac{2}{27}$ yards surface, 150 yards $19\frac{1}{3}$ feet solidity.

3. Required the surface of a vaulted roof, the length being 125 feet, and the girt 36 feet. Ans. 500 square yards surface.

Prob. II. To find the concavity of a vault.

Find the area of one of its ends according to its form, whether circular, elliptical, or Gothic, and multiply it by the length of the vault.

1. Required the content of a semi-circular vault, the span being 30 feet, and the length 150 feet.

Ans. 53014·5 cubic feet.

2. Required the content of an oval vault, the span being 30 feet, the height 12, and the length 60 feet.

Ans. 16964·64 cubic feet.

3. Required the vacuity of a Gothic vault 20 feet long, the span 50 feet, the chord of each of the arches 50 feet, and the versed sine of the arch 15 feet. Ans. 43024·2 cubic feet.

OF GROINS.

Groins are formed by the intersection of vaults with one another.

Prob. I. To find the surface of a groin.

Divide the area of the base by 7, and add the quotient to the dividend: the sum will be the area.

1. Required the surface of a groin raised upon a square, of which each side is 14 feet. Ans. 224 square feet.

2. Required the surface of a groin raised upon a rectangular base, of which the sides are 14 and 18 feet.

Ans. 288 square feet.

3. Required the surface of a circular groin-arch raised on a square base, each side 20 feet. Ans. $457\frac{1}{7}$ square feet.

Prob. II. To find the vacuity of a groin.

Multiply the area of the base by the height, and from the product subtract $\frac{1}{10}$ of it: the remainder will be the solidity.

Note. Instead of subtracting $\frac{1}{10}$ of the product, it may be multiplied by ·9, or by ·904.

1. Required the vacuity of a circular groin upon a square base, of which the side is 14 feet, and its height 7 feet.

Ans. $14^2 \times 7 - 14^2 \times \cdot 9 = 1234\frac{4}{5}$ cubic feet.

2. Required the vacuity formed by an elliptical groin, the side of its square base being 28 feet, and its height 9 feet.

Ans. $6350\frac{2}{5}$ cubic feet.

3. Required the vacuity of an elliptical groin upon a rectangular base 20 feet by 30, and the height 12 feet.

Ans. 6480 cubic feet.

OF DOMES.

A Dome is formed by arches springing from a circular or polygonal base, and meeting in a point at the top.

Prob. I. To find the surface of a dome.

Multiply twice the area of the base by the height; and the product, divided by the radius of the base, will give the surface.

1. Required the surface of a spherical dome upon a hexagonal base, of which the side is 10 feet.

Note. The radius of the base being equal to the height, twice the area of the base is the surface, $= 519 \cdot 615$ square feet.

2. Required the surface of a dome 20 feet high, upon a circular base, of which the circumference is 100 feet.

Ans. 2000 square feet.

3. Required the expense of painting a spherical dome upon an octagonal base, of which the side is 20 feet, at 8d. per square yard. Ans. £14, 6s. $1\frac{1}{2}$d.

Prob. II. To find the vacuity of a dome.

Multiply the area of the base by two-thirds of the height.

1. Required the content of a spherical dome, the diameter of its circular base being 30 feet.

Ans. $30^2 \times \cdot 7854 \times \frac{2}{3} \times 15 = 7068 \cdot 6$ cubic feet.

2. Required the solid content of an octagonal dome, of which the height is 21 feet, and each side of the base 20 feet.

Ans. 27039·19176 cubic feet.

3. Required the solid content of a dome upon a nonagonal base, of which the side is 12 feet, and the height 30 feet.

Ans. 17803·6537 cubic feet.

OF SALOONS.

Saloons are formed by arches connecting the side-walls of a building with a ceiling or platform in the middle.

PROB. I. To find the surface of a saloon.

Apply a line close to the arch, across the surface, from the side-wall to the platform, for its breadth, and measure along the middle of it quite round the room for its length, and multiply one of these by the other, to get the surface.

1. The girt across the face of a saloon is 4 feet, and the mean length round the room is 108 feet. Required the surface.
Ans. 432 square feet.

2. The girt across the face of a saloon is 7 feet 10 inches, and the mean length round the room 140 feet. What will the plastering of it cost, at $6\frac{3}{4}$d. per square yard, and the painting in oil, at 15d. per square yard?
Ans. £3, 8s. $6\frac{1}{2}$d. plastering; £7, 12s. $3\frac{3}{4}$d$\frac{1}{9}$ painting.

3. The mean length of a saloon is 127 feet 6 inches, and the breadth across the face of the saloon 6 feet. What will the size-painting of it cost, at $4\frac{1}{4}$d. per square yard?
Ans. £1, 10s. $1\frac{1}{4}$d.

PROB. II. To find the vacuity of a saloon.

Take the perpendicular height of the ceiling above the side-wall, and the horizontal distance between them, and multiply the one by half the other. Again measure a straight line from the top of the side-wall to the edge of the ceiling, and take the distance of the arch from the middle of this line, and also the distance of the middle of the arch from the top of the side-wall, and to $\frac{4}{3}$ of this distance add the straight line from the side-wall to the platform, and multiply the sum by $\frac{2}{5}$ of the distance of this last line from the arch. Subtract this product from the former, and multiply the remainder by the mean length round the room, taken as before. This will give nearly the part cut off by the saloon. Subtract this from the whole vacuity of the room, supposing the wall to go upright as high as the ceiling: the difference will be the vacuity.

Suppose the perpendicular height of a saloon to be 38·4 inches, the horizontal distance from the platform to the side-wall 37·9 inches, the chord of the arch 54 inches, and the distance of its middle point from the arch 9 inches, the chord of half the arch 28·44 inches, and the compass round the middle of the saloon 50 feet. Required the vacuity.

Ans. $(\frac{4}{3} \times 28{\cdot}44 + 54) \times \frac{2}{5} \times 9 = 330{\cdot}912$, and $37{\cdot}9 \times \frac{1}{2} \times 38{\cdot}4 = 727{\cdot}68$, and $727{\cdot}68 - 330{\cdot}91 = 396{\cdot}77$ square inches $= 2{\cdot}755$ square feet; therefore $2{\cdot}755 \times 50 = 137{\cdot}75$ cubic feet is the content occupied by the saloon, which, taken from the whole upright space, will leave the vacuity.

ON THE FLEXIBILITY, STRENGTH, AND FRACTURE OF TIMBER.

A piece of solid matter may be exposed to four distinct kinds of strains. 1st, It may be torn asunder, as in the case of ropes, tie-beams, king-posts, stretchers, &c. 2d, It may be crushed, as in the case of truss-beams, columns, posts, &c. 3d, It may be broken across, as in the case of joists, beams, &c. 4th, It may be twisted or wrenched, as in the case of axles of wheels, the nail of a press, &c.

The subjoined table of data, with the practical problems, have been deduced from a number of careful experiments made by Barlow, Tredgold, and others.

TABLE

OF THE FLEXIBILITY AND STRENGTH OF TIMBER.

Name of the kind of Wood.	Specific Gravity.	Value of U.	Value of E.	Value of S.	Value of C.
Teak, ...	745	818	9657802	2462	15555
Poon, ...	579	596	6759200	2221	14787
English oak, ...	969	598	3494730	1181	9836
Do. specimen 2, . . .	934	435	5806200	1672	10853
Canadian oak, . . .	872	588	8595864	1766	11428
Dantzic oak, ...	756	724	4765750	1457	7386
Adriatic oak, ...	993	610	3885700	1583	8808
Ash, ...	760	395	6580750	2026	17337
Beech, ...	696	615	5417266	1556	9912
Elm, ...	553	509	2799347	1013	5767
Pitch pine, ...	660	588	4900466	1632	10415
Red pine, ...	657	605	7359700	1341	10000
New England fir, . .	553	757	5967400	1102	9947
Riga fir, ...	753	588	5314570	1108	10707
Do. specimen 2, . . .	738	—	3962800	1051	—
Mar Forest fir, . . .	696	588	2581400	1144	9539
Do. specimen 2, . . .	693	403	3478328	1262	10691
Larch, ...	531	411	2465433	653	—
Do. specimen 2, . . .	522	518	3591133	832	—
Do. specimen 3, . . .	556	518	4210830	1127	7655
Do. specimen 4, . . .	560	518	4210830	1149	7352
Norway spar, ...	577	648	5832000	1474	12180

Prob. I. To find the strength of direct cohesion of a piece of timber of any given dimensions.

RULE. Multiply the area of the transverse section, in inches, by the value of C in the table, and the product will be the strength required in pounds.

NOTE. If the specific gravity differs from the mean tabular specific gravity, multiply the product by the specific gravity, and divide by the specific gravity in the table for the correct strength.

1. What weight will it require to tear asunder a piece of English oak, specimen 1, 4 inches square, the specific gravity being 969? Ans. 157376 lbs.

2. What weight will it require to tear asunder a piece of beech 3 inches square? Ans. 89208 lbs.

3. What weight will tear asunder a cylinder of red pine 6 inches in diameter? Ans. 282744 lbs.

PROB. II. To find the deflection of a beam fixed at one end, and loaded with any given weight at the other.

RULE. Divide 32 times the weight multiplied by the cube of the length of the beam in inches, by the continued product of the tabular value of E, into the breadth and cube of the depth of the beam, both being in inches.

NOTE. When the beam is loaded uniformly throughout, the rule still applies, only we multiply the cube of the length by 12 times the weight instead of 32 times.

1. If a weight of 300 lb. be hung upon the extremity of an ash batten 4 inches square, and projecting 5 feet from the wall where it is fixed, how much will it be deflected?

Ans. 1·23 inch.

2. How much would the same beam be deflected, if a prop proceeding from the wall met it at the distance of 2 feet from the wall? Ans. ·266 of an inch.

3. A batten of teak 10 feet long, 5 inches broad, and 6 inches deep, is fixed at one end, and a weight of 700 lbs. suspended from the other. Required its deflection, and also the deflection when loaded uniformly throughout its length.

Ans. 3·711 inches when the load is suspended from the end, and 1·3916 inches when disposed uniformly throughout.

4. A batten of Dantzic oak 20 feet long, 5 inches broad, and 6 deep, is fixed at one end, and loaded uniformly throughout with 1000 lbs. Required its deflection, and also the deflection when the load is suspended from the end, and the batten supported by a prop from the wall meeting it at 10 feet from the fixed end.

Ans. 32·23 inches in the first case, and 10·7433 inches in the second case.

PROB. III. To find the deflection of beams supported at both ends, and loaded in the middle with any given weight.

RULE. Divide the product of the cube of the length in inches by the given weight in lbs., by the continued product of the tabular value of E, into the breadth and cube of the depth in inches, for the deflection sought.

NOTE. When the beam is *fixed* at both ends, the deflection is ⅝ of that given in the rule.

1. A beam of pitch pine 8 inches broad, 3 thick, and 20 feet long, is supported at both ends, and loaded in the centre with a weight of 100 lbs. Required its deflection.

Ans. 4·408 inches.

2. A beam of Mar Forest fir, specimen 1, 14 inches broad, 9 deep, and 20 feet long, is supported at both ends. How much will it be deflected with 3000 lb. suspended at its centre?

Ans. 1·574 inch.

3. A beam of Canadian oak 6 inches broad, 8 deep, and 20 feet long, is fixed at both ends in a wall, and loaded at the centre with 4000 lbs. Required its deflection.

Ans. 4·71 inches.

PROB. IV. To find the deflection of beams supported at both ends, and loaded uniformly throughout their lengths with a given weight.

RULE. Multiply the deflection found by last problem by 5, and divide the product by 8, and the quotient will be the answer.

1. A beam of Norway spar 4 inches broad and 5 deep, is supported at both ends, the length being 20 feet. What will be the deflection when it is loaded uniformly throughout its length with a weight of 600 lbs. Ans. 1·777 inch.

2. A beam of English oak, specimen 1, 9 inches square and 20 feet long, supports a load of 3000 lbs. disposed uniformly throughout its length. Required the deflection.

Ans. 1·13 inch.

3. A beam of larch, specimen 3, 10 inches broad and 1 foot deep, supports the building over a gateway 10 feet wide. What deflection may be expected, supposing the whole weight 50,000 lbs.? Ans. ·742 of an inch.

PROB. V. To find the ultimate deflection of beams or rods supported at both ends, before their fracture.

RULE. Divide the square of the length in inches by the product of the tabular value of U, multiplied by the depth

the beam in inches, and the quotient will be the ultimate deflection.

1. A rod of poon, 2 inches square and 10 feet long, is broken by a weight applied to its centre. Required the deflection at the instant of fracture. Ans. 12·08 inches.

2. Required the ultimate deflection of a beam of Adriatic oak 6 inches square and 30 feet long. Ans. 35·41 inches.

3. Required the ultimate deflection of a beam of ash 1 foot broad, 8 inches deep, and 40 feet long. Ans. 72·91 inches.

PROB. VI. To find the ultimate transverse strength of any rectangular beam of timber fixed at one end and loaded at the other.

RULE. Multiply the tabular value of S by the breadth and square of the depth, both in inches, and divide the product by the length in inches, and the quotient will be the weight in pounds.

1. What weight will it require to break a piece of Riga fir, specimen 1, fixed by one end and loaded at the other, the breadth being 3 inches, the depth 4 inches, and 5 feet long?

Ans. $886\frac{2}{5}$ lbs.

2. What weight will it require to break a piece of ash fixed by one end and loaded at the other, the breadth being 6 inches, the depth 4 inches, and 7 feet long? Ans. $2315\frac{5}{7}$ lbs.

3. What weight uniformly distributed throughout the length of a beam of English oak, 2d specimen, will break it, the breadth being 6 inches, the depth 9 inches, and its projection from the wall in which it is fixed, 12 feet. Ans. 11286 lbs.

PROB. VII. To find the ultimate transverse strength of any rectangular beam when supported at both ends and loaded in the centre.

RULE. Multiply the tabular value of S by 4 times the breadth and square of the depth in inches, and divide the product by the length in inches for the weight.

NOTE 1. When the beam is *fixed* at each end, and loaded in the middle, the result obtained by the rule must be increased by its half.

NOTE 2. When the beam is loaded uniformly throughout its length, the result obtained by the rule must be doubled.

NOTE 3. When the beam is *fixed* at both ends, and loaded uniformly throughout, the result obtained by the rule must be multiplied by 3.

1. What weight will it require to break a beam of English oak, 2d specimen, supported at both ends and loaded in the

middle, the length being 12 feet, the breadth 6 inches, and the depth 8 inches? Ans. 17834$\frac{2}{5}$ lbs.

2. What weight will it require to break a piece of larch, 3d specimen, supported at both ends and loaded in the middle, the length being 8 feet 4 inches, the breadth 8 inches, and the depth 10 inches? Ans. 36064 lbs.

3. What weight will it require to break a beam of New England fir, fixed at both ends, and loaded uniformly throughout its length, which is 10 feet, and 6 inches square?

Ans. 23803$\frac{1}{5}$ lbs.

4. What weight will it require to break a beam of Riga fir, 1st specimen, fixed at both ends, and loaded at the centre, the length being 15 feet, the breadth 9 inches, and the depth 1 foot. Ans. 47865$\frac{3}{5}$ lbs.

NOTE. In Barlow's Essay on the Strength of Timber, a second rule is given to each of the two last problems, the angle of deflection being taken into consideration, which gives a greater result. The rules given here are, however, best for practice, as they are simpler, and two-thirds of their results for a permanent load is reckoned sufficient.

PROB. VIII. To find the weight under which a given column will begin to bend when placed vertically on a horizontal plane.

RULE. Multiply the tabular value of E by the cube of the least thickness, and by the greatest thickness, both in inches, and that product again by ·2056. Then divide the last product by the square of the length in inches for the weight in pounds.

1. What weight will it require to bend a column of ash 4 inches square and 6 feet 8 inches long, when placed vertically on a plane, and the weight applied at its upper extremity?

Ans. 54120·088 lbs.

2. What weight will it require to bend a column of English oak, specimen 2, 20 feet long, 6 inches thick, and 9 broad?

Ans. 40289·222 lbs.

3. What weight will it require to bend a column of Riga fir, specimen 1, 15 feet long, and 10 inches in diameter?

Ans. 337245·553 lbs.

4. What weight will it require to bend a column of New England fir 20 feet long, and 1 foot in diameter?

Ans. 441683·08 lbs.

PROMISCUOUS QUESTIONS.

1. How many stones of a rectangular form, each 3 feet by $2\frac{1}{2}$ feet, will pave a road 40 yards long, and 6 yards broad?
Ans. 288 stones.

2. How many panes of glass, each 18 inches by 14 inches, will be required for 22 windows, each 5 feet by 3 feet 6 inches?
Ans. 220 panes.

3. What is the excess of a floor, 50 feet long by 30 broad, above two others, each of half its dimensions?
Ans. 750 square feet.

4. How much must be cut off from a board 26 inches broad, to contain $1\frac{1}{2}$ square yards. Ans. 6·23 feet.

5. The ceiling of a room 28 feet broad, contains 114 square yards 6 feet. What is the length of the room?
Ans. $36\frac{6}{7}$ feet.

6. Along one side of a court 47 feet 9 inches square, there is a footpath 4 feet broad. What will be the expense of laying the rest of it with stones, at 6d. per square yard?
Ans. £5, 16s. $0\frac{3}{4}$d.

7. A room is 60 feet in circuit, and 12 feet high. How much paper, 2 feet wide, will line it, deducting the door, 8 feet by 4 feet, and 3 windows, each 5 feet by $3\frac{1}{2}$ feet, and the chimney 4 feet square? Ans. $103\frac{1}{4}$ yards.

8. The base of a right-angled triangle is 300 feet, and the sum of the other two sides is 1000 feet. What are their lengths? Ans. 545 and 455 feet.

9. A roof which is 24 feet 8 inches, by 14 feet 6 inches, is to be covered with lead, at 8 lbs. to the square foot. Required the expense, at 2 guineas per cwt.? Ans. £53, 13s.

10. How many square feet of deal will be required to make a rectangular chest, of which the length is to be $3\frac{1}{2}$ feet, the breadth 2 feet, and the depth 20 inches?
Ans. $32\frac{1}{3}$ square feet.

11. A beam is $8\frac{1}{2}$ inches deep and $3\frac{1}{2}$ feet broad. Required the depth of another twice as large, which is $4\frac{3}{4}$ inches broad?
Ans. 12·526 inches deep.

12. The four sides of a trapezium are, 13, 13·4, 24, and 11 feet, and the two first contain a right angle. Required the area. Ans. 253·38 square feet.

13. What will be the expense of paving a semi-circular plot, of which the diameter is 14·8 feet, at 2s. 4d. per square foot? Ans. £10, 0s. 8¼d.

14. The wheels of a chaise, each 4 feet high, in turning within a ring, moved so that the outer wheel made two turns while the inner made one, and their distance from one another was 5 feet. What were the circumferences of the tracks described by them?

Ans. Outer, 62·8318 feet. Inner, 31·4159 feet.

15. A circular pond occupies half an acre. What was the length of the cord which struck the circle? Ans. 27¾ yards.

16. A right-angled triangle has its base 16, and its perpendicular 12, and a triangle is cut off from it by a line parallel to the base, of which the area is 24. What are the lengths of the sides of that triangle? Ans. 8, 6, and 10.

17. An ellipse is surrounded by a wall 14 inches thick, its axes are 840 links and 612 links. Required the quantity of ground enclosed, and the quantity occupied by the walls.
Ans. 4 ac., 6 perches enclosed, and 1760·4933 sq. feet of wall.

18. What is the length of the side of an equilateral triangle, which cost as much for paving the area of it, at 8d. per square foot, as for pallisading its 3 sides at a guinea per lineal yard? Ans. 72·746 feet.

19. How long must be the tether of a horse which will allow him to graze quite round an acre of ground? Ans. 39¼ yards.

20. How many 3 inch cubes may be cut out of a 12 inch cube? Ans. 64 cubes.

21. What will be the expense of painting a conical spire, of which the height is 118 feet, and the circumference of the base 64 feet, at 8d. per square yard? Ans. £14, 0s. 8·96d.

22. The diameter of a standard bushel is 18½ inches, and its depth 8 inches. What must be the diameter of that bushel which is 7½ inches deep? Ans. 19·1067 inches.

23. What will be the expense of gilding a globe, of which the diameter is 6 feet, at 3½d. per square inch? Ans. £237, 10s. 1·19d.

24. A farmer borrowed a cubical piece of hay, which measured 6 feet every way, and he repaid two cubical pieces, of

which the sides were 3 feet each. What part of the quantity borrowed has he returned? Ans. The fourth part only.

25. A person wants a cylindrical vessel 3 feet deep, which shall hold twice as much as another 28 inches deep, and 46 inches in diameter. What must be the diameter of the required vessel? Ans. 57·373 inches.

26. What will be the diameter of a globe, of which the superficial and solid contents are both expressed by the same number? Ans. 6.

27. A sack $22\frac{1}{2}$ inches broad when empty, will contain 3 bushels of corn when filled. What will another sack contain, which is twice its breadth, and of the same length?

Ans. 12 bushels.

28. A cable 3 feet long, and 9 inches in circuit, weighs 22 lbs. What will be the weight of a fathom of that cable, of which the circumference is a foot? Ans. $78\frac{2}{9}$ lbs.

29. The distance between the centres of two circles, each 50 feet diameter, is 30 feet. What is the area of the space enclosed by their circumferences? Ans. 559·119 square feet.

30. What is the length of the chord which cuts off $\frac{1}{3}$ of the area from a circle, of which the diameter is 289 feet?

Ans. 278·6538 feet.

31. A sugar-loaf in form of a cone is 20 inches high, it is required to divide it equally among three persons by sections parallel to the base. What is the height of each part?

Ans. Upper 13·8672, next 3·6044, lowest 2·5284 inches.

32. A malt-kiln is $16\frac{1}{2}$ feet square. Required the side of a square kiln, which is capable of drying three times as much malt. Ans. 28·5788 feet.

33. A round cistern is 26·3 inches in diameter, and $52\frac{1}{2}$ inches deep. What should be the diameter of another of the same depth to contain twice the quantity of liquor?

Ans. 37·1938 inches.

34. How many rafters, each $2\frac{1}{2}$ inches by $1\frac{1}{2}$ inches, can be sawed out of a square log $17\frac{1}{2}$ inches by 10 inches?

Ans. $46\frac{2}{3}$ rafters.

35. How many bricks, each 9 inches long, $4\frac{1}{2}$ inches broad, and 3 inches thick, must be taken to build a wall 100 feet long, 20 feet high, and one foot thick? Ans. $28444\frac{4}{9}$ bricks.

36. A piece of round timber, containing 20 solid feet, is to be hewn into square timber. How much will it contain when squared? Ans. 12·732 solid feet.

37. What must be the dimensions of a cubical chest to hold 200 oranges, each $2\frac{1}{2}$ inches in diameter?

Ans. Each side 14·62 inches.

38. When the price of timber is 16d. per running foot, 14d. per superficial foot, and 20d. per solid foot, which of these is best for the seller, and what will he gain upon a plank 14 feet long, $1\frac{1}{2}$ feet broad, and 6 inches thick?

Ans. 224d. value running, 210d. solid, and 294d. superficial.

39. A board is 10 feet long, 8 inches in breadth at the greater end, and 6 inches at the less. How much must be cut off from the less end to make a square foot?

Ans. 23·2493 inches.

40. If a cubic foot of brass be drawn into wire of $\frac{1}{10}$ inch diameter, what will be the length of the wire, supposing no loss of metal in working?

Ans. 97784·5684 yards, or nearly 56 miles.

41. How high above the earth must a man be raised to see $\frac{1}{3}$ of its surface? Ans. One diameter high.

42. A frustum of a cone of marble has its slant side 8 feet, and the diameters of its bases 4 feet and 1·5 feet. What is its value at 12s. per solid foot? Ans. £30, 1s. 11½d.

43. A garden is 100 feet long and 80 feet broad, and a border of equal breadth surrounds the sides of it, which is just ½ of the garden. What is the breadth of the border?

Ans. 25·9688 feet.

44. A carpenter put a curb of oak round a well: the inner diameter of the curb was $3\frac{1}{2}$ feet, and its breadth $7\frac{1}{2}$ inches. What was the expense of it at 8d. per square foot?

Ans. 5s. 2¼d.

45. A piece of square timber is 10 feet long, each side of the greater base 9 inches, and each side of the less 6 inches. How much must be cut off from the less end to contain a solid foot? Ans. 3·392 feet.

46. The girt of a vessel round the outside of the hoop is 22 inches, and the hoop is 1 inch thick. What is the true girt of the vessel? Ans. 15⅝.

47. Required the superficial and the solid contents of an elliptical ring in form of a cylinder, the inner diameters of the ellipse being 38 and 28 inches, and the thickness of the metal in the ring 2 inches?

Ans. 694·3826 square inches in surface, 347·1913 cubic inches solidity.

48. Required the axis of the greatest cone which can be cut out of a globe, of which the axis is 30 inches.

Ans. 20 inches the axis, and 28·28427 inches the diameter of its base.

49. Four men bought a grinding stone of 30 inches in diameter, and agreed that the first should use it till he ground down $\frac{1}{4}$ of it for his share, deducting 6 inches of diameter in the middle for waste, and then that the second should use it till he ground down another $\frac{1}{4}$ part, and so on. What part of the diameter must each grind down for his share?

Ans. The 1st 3·8466 inches, 2d 4.5201 inches, 3d 5·7588 inches, 4th 9.8745 inches.

50. Given the distance 12 between the focus of an ellipse and the nearest principal vertex, and the ratio of the curve as 4 to 5, to find the area of the ellipse. Ans. 6785·856.

51. Required the area of a parabola, of which the axis is 120, and the distance of the focus from the principal vertex 10·3, or the perimeter 43·2. Ans. 11520.

52. A gentleman has a bowling-green 300 feet long, and 200 feet broad, which he wishes to raise a foot higher by means of the earth dug out of a ditch which surrounds it. To what depth must the ditch be dug, supposing its breadth to be 8 feet? Ans. $7\frac{25}{86}$ feet.

53. Of what diameter must a piece of ordnance be, which is cast for a ball of 24 lbs. weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball?

Ans. 5·6918 inches.

54. Suppose the windage of a mortar to be $\frac{1}{60}$ of the diameter of the mortar, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of that of the mortar. It is required to determine the diameter and weight of the shell, and the weight of the powder requisite for the mortars in common use, viz. those of 13, of 10, of 8, of 5·8, and of 4·6 inches in diameter.

Ans. The diameters of the shells are $12{\cdot}78\dot{3}$, $9{\cdot}8\dot{3}$, $7{\cdot}8\dot{6}$, $5{\cdot}70\dot{3}$, and $4{\cdot}52\dot{3}$ inches. Their weights are 183·3, 83·43, 42·72, 16·28, and 8·12 lbs., and the weights of the powder 13·15, 5·99, 3·065, 1·168, and 0·58 lbs.

55. How many shot are in a triangular pile, of which a side of the base contains 50? Ans. 22100 balls.

56. How many shot are in an oblong pile, of which the sides of the base contain 49 and 19? Ans. 8170 balls.

57. How many shot are in an unfinished triangular pile, each side of the bottom being 50 and of the top 20? Ans. 20770 balls.

58. How many shot are in an incomplete oblong pile, the length and breadth of the base being 50 and 20, and the length and breadth at the top 38 and 8? Ans. 8190 balls.

59. Required the weight of lead in a pipe 600 yards long, the diameter of the bore being $1\frac{1}{4}$ inches, and the thickness of the metal $\frac{1}{4}$ inch. Ans. 10448.2744 lbs.

60. Required the content of a frustum of a cone, of which the greatest diameter is 60 inches, the diagonal between the farthest extremities of the diameters 66, and the slant side 30 inches. Ans. 293.61 imp. gallons.

61. If a heavy sphere, of which the diameter is 4 inches, is dropt into a conical glass full of water, of which the diameter is 5 inches, and the altitude 6 inches, How much water will run over? Ans. 26.27215 cubic inches.

62. Suppose it is found that a ship, with its ordnance, rigging, &c. displaces 50,000 cubical feet of water, What is the weight of the vessel? Ans. 1395·0893 tons.

63. If a solid inch of metal weighs 8 ounces avoirdupois, What is its specific gravity? Ans. 13824.

64. If a man weighs 192 lbs., and the specific gravity of his body be 1200, How much cork must be tied to him to make him swim? Ans. $10\frac{2}{19}$ lbs.

65. If a cube of solid fir, 12 inches each way, sinks 6 inches in water, What is its specific gravity? Ans. 500.

66. Four solid inches of copper is to be made into a hollow cube. How thick must the metal be that it may swim in one inch depth of water? Ans. ·01863 inches.

67. If two solid feet of feathers weigh 4 lbs., What will the same quantity weigh when compressed into the bulk of half a solid foot, supposing a solid foot of air to weigh $1\frac{1}{5}$ oz.?
Ans. 4 lbs. 1·8 oz.

68. If a man standing at the side of a river hears his voice reflected from the opposite bank in 3 seconds of time, What is the breadth of the river? Ans. 1713 feet.

69. I saw the flash of a gun fired from a ship at sea, and 33 seconds afterwards I heard the report. How far was the ship distant from me? Ans. $7\frac{11}{80}$ miles.

70. Observing a battery of cannon, I counted 17 seconds on my watch between the times of seeing the flash and of hearing the report. How far was I distant from the battery?
Ans. $3\frac{1783}{2640}$ miles.

71. The frustum of a cone is 5·7 inches in height, the dia-

meter at the top 3·7 inches, and that at the bottom 4·23 inches. Required the difference between the contents of the hoofs into which it is divided by a plane passing through the opposite extremities of its diameters. Ans. 7·0532 cubic inches.

72. Required the contents of the hoofs into which a cone of which the height is 6 inches, the top diameter 3, and the bottom diameter 4 inches, is divided by a plane passing from the edge of the top to the centre of the base.
Ans. The less hoof 15·2628, the greater 42·8568 cubic inches.

73. Suppose a cubic inch of common glass to weigh 1·4921 oz. avoirdupois, one of sea-water ·59542 oz., and one of brandy ·5368 oz. How much force will be required to buoy up in the sea an imperial gallon of brandy in a bottle, of which the weight of the glass in air is 3·84 lbs.? Ans. 20·669 oz.

74. How far will a body descend from a state of rest in 20 seconds? Ans. $6433\frac{1}{3}$ feet.

75. If a body is projected perpendicularly in free space with a velocity of 10,000 feet per second, To what height would it ascend, and in what time would it again reach the earth?
Ans. $294\frac{2513}{6369}$ miles, and in $621\frac{147}{193}$ seconds.

76. Suppose that at the moment a body is projected up AB with the velocity acquired by falling down it, another body begins to fall down it, In what point will they meet, AB being $1029\frac{1}{3}$ feet? Ans. 772 feet from the bottom.

77. Suppose that a body is projected downwards with a velocity of $64\frac{1}{3}$ feet per second, and in 2 seconds after another body is projected down with a velocity of $258\frac{1}{3}$ feet, In what time will it overtake the other? Ans. $1\frac{1}{2}$ second.

78. A person from a window 20 feet high observes in a mirror placed 12 feet from the foundation of the house the top of a spire 100 feet high. Required the distance of the observer from the spire. Ans. 72 feet.

79. Melville's Monument in St Andrew's Square, Edinburgh, is 136 feet 4 inches high, and the statue on the top 14 feet high. At what distance from the base of the monument does the statue subtend the greatest angle?
Ans. 143·1622 feet.

80. Two trees, 100 feet asunder, are placed, the one at the distance of 100 feet, and the other 50 feet from a wall. What is the shortest distance that a person must pass over in running from one tree to touch the wall, and then to the other tree?
Ans. 171·334 feet.

81. I took two stations A and B at the distance of 150 feet

from each other, and in the same straight line with an inaccessible spire; then from A, the station nearest the spire, in a line perpendicular to the line AB, I measured AC 160 feet, and set up a pole at the extremity C; and from B, the other station in a line also perpendicular to AB, I measured the distance BD 275·5 feet, when I observed that the spire and the pole at C were in the same straight line with the point D. Required the distance of the spire from the station A.

Ans. 207·79 feet.

82. What is the weight of a sphere of oak 6 feet in diameter, its specific gravity being 925? Ans. 2·91895 tons.

83. To what depth would a cube of beech 2 feet 6 inches in the side sink in water? Ans. 2·13 inches.

84. A horse's tether of 40 yards in length is fixed in the circumference of a circular field whose diameter is 350 yards. How much will it allow him to graze? And, supposing that the end of the tether is removed to the circumference of the secondary circle, and in a line with the centre of the field, What additional space would he be enabled to graze?

Ans. First 2391·2695 square yards; and afterwards 3061·1712 square yards.

85. The axes of a punch-bowl in the form of the segment of an oblong spheroid are to each other as 3 to 4, the depth is $\frac{1}{4}$ of the longer axis, and the diameter of its top is 20 inches. What number of rounds may a company of 30 persons drink out of it, using a conical glass of which the top diameter is $1\frac{1}{2}$ inches, and the depth 2 inches? Ans. 38·01499 rounds.

86. A certain island is 73 miles in circumference, and if 2 men set out from the same point in the same direction, the one travelling at the rate of 5 and the other at the rate of 3 miles an hour, In what time will they be together again?

Ans. $36\frac{1}{2}$ hours.

87. Required the solidity of the greatest cone which can be cut out of an oblong spheroid of which the axes are 40 and 60 inches.

Ans. 22340·26 feet when the axis of the cone is in the minor axis of the spheroid, and 14893·51 when the axis of the cone is in the major axis.

88. Suppose a cone 20 feet high, and the diameter of the base 6 feet, is cut through the axis 5 feet from the bottom, at an angle of 60 degrees. Required the solidity of the sections.

Ans. Solidity of the upper 82·296 feet. Solidity of the under 106·2 feet.

APPENDIX.

SECTION I.

GENERAL PRINCIPLES OF GEOMETRY.

THE demonstrations of many of the rules given in Trigonometry and Mensuration were judged too long to be inserted in the text; they are, therefore, added here, and to them are prefixed the general principles of geometry upon which they depend.

A straight line may be drawn between two points, by laying a *ruler* or another straight line by these points, and tracing a line along the side of it.

But the only original method of producing a straight line is, by stretching a hair or thread through the two points; and as the thread assumes invariably the same position as often as it is stretched through the same points, and a less portion of it lies between the points when it is stretched, than when it lies loosely between them, it follows,

First, That a straight line between two points has only one position.

Secondly, That both sides of a straight line are exactly alike.*

Thirdly, That a part of a straight line is in every respect similar to another part of it, or to another straight line of the same length.

Fourthly, That the straight line is the shortest distance from one point to another.

From these properties of a straight line it is inferred,

1st, That two straight lines will coincide when they are applied to one another, in what way soever the application is made.

2d, That one straight line cannot cut another in more points than one.

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* If a hair stretched between the points A and B coincide with the trace AB, and if then the part of it at A be brought to B, and that at B to A, so that the upper side of it may now be the lower one, the stretched hair will again coincide with the trace AB.

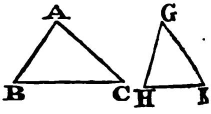
A————B

3d, And consequently that two straight lines can neither have a common segment nor enclose a space.

4th, That a straight line is less than a curve, or than the sum of any number of straight lines joined together, which terminate at the same points with it.

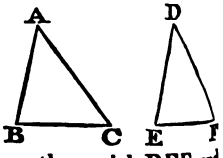
5th, That straight lines which have the same position, in respect of the same straight line, must either coincide or be parallel to one another.*

PROPOSITION I. Two triangles ABC, GHK are equal in every respect, when an angle BAC and the two sides AB, AC, which contain it in one of them, are respectively equal to an angle HGK, and the sides GH, GK containing it in the other.

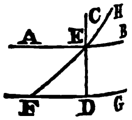


For, if the triangle ABC lie on GHK, so that A be on G and AB on GH, then AC will lie along GK, for the angle A = G, and B will be on H, and C on K; therefore BC will coincide with HK, the triangle ABC with GHK, the angle B with H, and C with K. They are all therefore equal.

PROP. II. If a side AB, and the two adjacent angles at A and B of one triangle ABC, be equal to a side DE, and the adjacent angles at D and E of another, the triangles are in all respects equal.



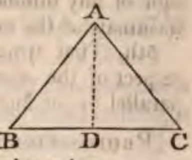
For, if the triangle ABC be laid on DEF, A on D, and AB on DE, then B will be on E, AC on DF and BC on EF, because the angles at A and B are equal to those at D and E; therefore, the angle C shall be on F, and the triangle ABC will coincide altogether with DEF, and be equal to it.



* If the straight lines AB and CD intersect in E, the angle CEB shows their relative situations; and these situations would remain though they should intersect in any other point of CD, as at D; in which case AB would become FG, and EC would coincide with DE. Of course, if the angle EDG be equal to CEB, the lines AB and FG would have the same direction, and if they have the same direction, the angle EDG would be equal to CEB; and for the same reason the angle HEB would be equal to EFD.

These things seem to follow immediately from the definitions of a straight line and of an angle, and, if admitted as principles, they would render several parts of geometry easy, which are at present difficult.

Prop. III. In an isosceles triangle ABC, the angles at B and C, opposite to the equal sides AC and AB, are equal to one another.



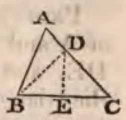
Bisect the angle BAC by AD, then the triangles ABD, ACD, have AB = AC, AD common, and the angle BAD = CAD; therefore, they are equal in every respect, (1.) and have the angle ABC = ACB.

Cor. 1. An equilateral triangle is also equiangular.

Cor. 2. The straight line AD which bisects the angle BAC, bisects also BC at right angles, and conversely.

Cor. 3. Two right-angled triangles, ADB and ADC, which have equal hypotenuses AB = AC, and an oblique angle DAB = DAC, are equal in every respect. For, supposing their perpendiculars to coincide in AD, the straight line BC which joins the extremities of AB, AC will be bisected at right angles by AD.

Prop. IV. The greater side AC of a triangle ABC has the greater angle ABC opposite to it.



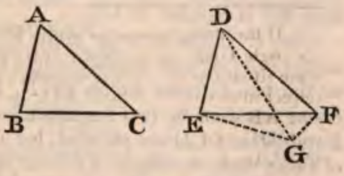
Bisect BC in E, draw ED perpendicular to BC* and join BD. The triangles BED, CED are equal, for BE = EC, ED common, and the angles at E are equal; therefore, the angle DBC = DCB, and the angle ABC $>$ ACB.

Cor. 1. If the angle ABC be greater than ACB, the side AC will be greater than the side AB.

Cor. 2. If the angle DBC be = DCB, then DC = DB.

Cor. 3. An equiangular triangle is also equilateral.

Prop. V. If two triangles ABC, DEF, have their three sides equal, each to each, the angles which are opposite to the equal sides will be equal.

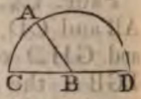


Let DE be the least side, and if possible let the angle BAC be less than EDF; and if AB be on DE, A on D, and so B on E, then AC will fall within the angle EDF as

* DE must first meet the greater side AC for (pr. 1.) DC = DB and AC = BD + DA, which by the 4th property of straight lines is greater than AB.

on DG, and BC on EG. Join FG, then the angle EGF = EFG, (3.) because EF = EG; but DGF, a part of the first, is equal to DFG, for DG = DF, which is greater than the other.* As this cannot be, the angle BAC must be equal to EDF, and ABC = DEF and ACB = DFE.

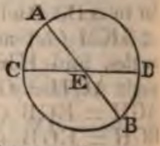
Prop. VI. The adjacent angles ABC and ABD on the same side of the straight line CD, make together two right angles, or 180°.



For their measuring arcs AC and AD make $\frac{1}{2}$ of the circumference or 180°.

Cor. On the contrary, if the angles ABC, ABD make together 180°, CB and BD are in a straight line.

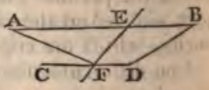
Prop. VII. The vertical angles AEC and BED, made by two straight lines AB and CD, which cut in E, are equal to one another.



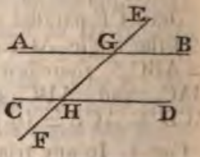
For the arcs CAD and ADB being each $\frac{1}{2}$ of the circumference are equal; therefore, AC = BD, and the angle AEC = BED.

Cor. All the angles about a point are together equal to four right angles.

Prop. VIII. If a straight line EF meet two straight lines AB and CD, and make the alternate angles AEF, EFD equal to one another, these two straight lines AB and CD are parallel.



If not, let them meet if possible in B, and make AE = BF, and join AF. Because AE = FB and EF common to the triangles AEF, BFE, and the angle AEF = BFE, the triangles are equal, (Prop 1.) and the angle AFE = BEF, and the two angles AFE + EFB = AEF + BEF = two right angles (6.), therefore AF and FB are in a straight line, which cannot be (Gen. Prop. 3.); therefore AB is parallel to CD.



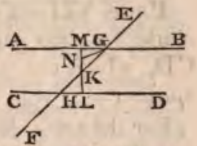
Cor. 1. If the *exterior* angle EGB be = the *interior* and opposite angle EHD, or the two *interior* angles BGF, EHD equal together to two right angles, the lines AB and CD are parallel, for in each of these cases the angle AGF = EHD.

* The point G cannot fall within the triangle DEF, for then DFE being equal to DFG + EFG, would be equal to DGF + EGF, which is greater than two right angles.

Cor. 2. Straight lines AB, CD perpendicular to the same straight line EF are parallel, for the right angles AGF, EHD are equal.

Assumption. If two straight lines be parallel, a straight line which is perpendicular to one of them is also perpendicular to the other.

Prop. IX. If a straight line EF cut two parallels AB and CD, it will make the alternate angles AGH and GHD equal to one another, the exterior angle EGB = the interior and opposite GHD, and the two interior angles BGH, GHD, on the same side of it, equal to two right angles.



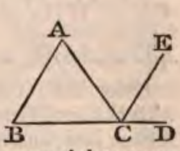
Bisect GH in K, and draw KL perpendicular to CD, it is also perpendicular to AB. And because the angle HKL = GKM (7.) and HLK, KMG right angles, and HK = KG; therefore, the angle AGH = GHD (3. Cor. 3.). Also AGH = EGB (7.) therefore EGB = GHD and GHD + BGH = EGB + BGH = two right angles.

Cor. 1. If the two interior angles be less than two right angles, the straight lines will meet if produced far enough.

Cor. 2. A straight line which meets one of two parallels will, if produced, meet the other also.

Scholium. When a straight line meets two parallels, the angles are equal, which are either on the same side of it, and also of the parallels, or on different sides both of it and of the parallels. And the two angles are together equal to two right angles, which are either on the same side of the cutting line, and on different sides of the parallels, or on different sides of it, and on the same side of the parallels.

Prop. X. The exterior angle ACD of a triangle is equal to both the interior and opposite angles ABC + BAC, and the three angles ABC + BAC + ACB, are together equal to two right angles.



Draw CE parallel to AB; it will make (9.) the angle ACE = BAC, and ECD = ABC; therefore, ACD = ABC + BAC, and ABC + BAC + ACB = ACD + ACB = (6.) to two right angles.

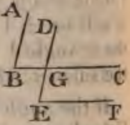
Cor. 1. In any triangle, there can be only one right or one obtuse angle.

Cor. 2. In a right-angled triangle, the two acute angles are together equal to a right angle.

Cor. 3. An angle of an equilateral triangle is two-thirds of a right angle, or it is 60°.

Cor. 4. When two angles of a triangle are known, the third angle is got by subtracting their sum from 180°.

Prop. XI. If two angles ABC, DEF have their sides parallel, and in the same direction, they are equal.



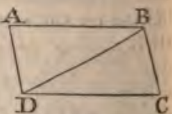
Let DE, produced if necessary, meet BC in G. Then the angle ABC = DGC, and DGC = DEF (9.); therefore, the angle ABC = DEF.

Prop. XII. All the exterior angles FAB, GBC, &c. of any rectilineal figure, are together equal to four right angles, or 360°.

Draw AM parallel to BC, AN parallel to CD, AP to DE. Then the angle GAM = GBC, MAN = HCD, &c. (9.) Therefore, all the exterior angles are equal to the angles about the point A, that is, to four right angles (7. Cor.).

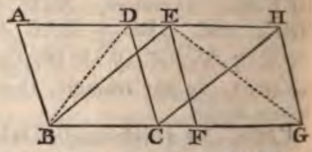
Cor. Since each interior angle, with its adjacent exterior, makes two right angles (6.), all the interior angles, together with four right angles, make twice as many right angles as the figure has sides. Thus the interior angles of a quadrilateral make 4 right angles, of a pentagon 6 right angles, of a hexagon 8, of a heptagon 10, &c.

Prop. XIII. The opposite sides and the opposite angles of a parallelogram ABCD are equal to one another, and the diagonal BD bisects it.



Since BD meets the parallels, it makes (9.) the angle BDC = ABD and DBC = ADB, and the side DB is common to the triangles ADB and DBC, they are therefore in all respects equal (2.).

Prop. XIV. Parallelograms ABCD, EFGH, upon equal bases BC = FG, and between the same parallels AH and BG, are equal to one another.



Draw BE, CH. Since AD=BC=FG=EH (13.) AE = DH, and AB = DC, and the angle HDC = EAB (9.); therefore, the triangle EAB = HDC (1.); take these equals from ABCH, and the remainder ABCD =

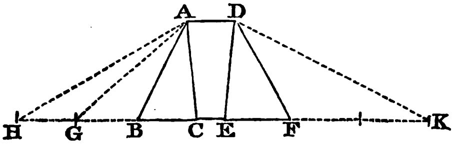
EBCH. For the same reason EFGH = EBCH; therefore, ABCD = EFGH.

Cor. Triangles DBC, EFG upon equal bases, and between the same parallels, are equal; for they are the halves of the parallelograms.

Scholium. If ABCD be a rectangle, it is = BC × AB, (Mens. Prob. 1.). Therefore, if EFGH be any parallelogram, it will be = FG × perpendicular between EH and FG. And the triangle EFG = $\frac{1}{2}$ FG × perpendicular on it, which are the rules in Mens. Surfaces, Prob. 2 and 4.

If the angle at F be given, the perpendicular = EF × Sin F (radius = 1), see Ex. 1. page 100. Therefore, the parallelogram EFGH = FG × EF × Sin F, and the triangle EFG = $\frac{1}{2}$ FG × EF × Sin F, which are the rules in Mens. Surfaces, Prob. 3. and 5.

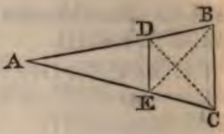
Prop. XV. Triangles ABC, DEF between the same parallels AD and BF, are to one another as their bases. BC : EF : : ABC : DEF.



Let CB, BG, GH be all equal, and n their number, so that CH = n × CB, and draw AG, AH, the triangles ABC, AGB, AHG are equal (14. Cor.), and therefore AHC = n × ABC. Take EK the least number of times EF, which is greater than CH, and let FK = m × EF, and draw DK, then the triangle DFK = m × DEF. And because CH or n × BC is not less than FK or m × EF, but less than EK or (m + 1) × EF, m is the quotient by which n × BC contains EF. And the triangle AHC or n × ABC is not less than DFK or m × DEF, but less than DEK, or (m + 1) × DEF, therefore m is also the quotient by which n × ABC contains DEF, so that n × BC divided by EF, and n × ABC divided by DEF give the same quotient. Wherefore BC : EF : : ABC : DEF.

Cor. Triangles and parallelograms of equal altitudes are to one another as their bases.

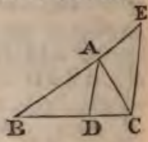
Prop. XVI. Parallels BC, DE, divide other straight lines proportionally. AD : DB . : AE : EC.



Draw BE and DC. The triangle BED=DEC (14. Cor.). Therefore ADE : DEB : : ADE : DEC. But (15.) AD : DB : : ADE : DEB and AE : EC : : ADE : DEC, therefore AD : DB : : AE : EC.

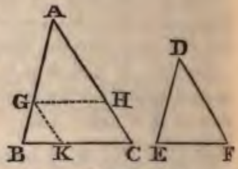
Cor. Straight lines which meet three parallels are cut proportionally by them.

Prop. XVII. If the angle BAC of a triangle ABC be bisected by AD, the segments of BC have the same ratio with the sides. BD:DC::BA:AC.



Draw CE parallel to AD. The angle BEC = BAD or = DAC ; that is, = ACE (9.) ; therefore AE = AC, and BD : DC : : BA : EA or AC (16.).

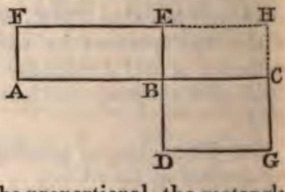
Prop. XVIII. Triangles ABC, DEF, which have two angles equal, each to each, A = D, and B = E, have their sides proportional.



Make AG = DE, and draw GH, GK parallel to BC, CA. The triangle AGH = DEF (2.) for AG = DE, the angle GAH = EDF and AGH = B = E; therefore AH = DF and GH = EF. But AB : AC : : AG : AH (16.) : : DE : DF. And AB : BC : : AG : KC = GH : : DE : EF.

Cor. If the sides be proportional, or if the sides about two equal angles be proportional, the triangles are equiangular.

Prop. XIX. If four straight lines be proportional AB : BC : : DB : BE, the rectangle contained by AB and BE, the extremes, is equal to that contained by DB and BC, the means.



Let DE be perpendicular to AC, and complete the rectangles AE, EC and CD. Then AE : EC : : AB : BC (15.) : : DB : BE : : DC:CE; therefore AE = CD.

Cor. 1. If three straight lines be proportional, the rectangle contained by the extremes is equal to the square of the mean.

Cor. 2. If the rectangle AE, contained by AB and BE the

extremes, be equal to the rectangle DC, contained by DB and BC the means, then AB : BC : : DB : BE.

Cor. 3. Any parallelogram or triangle contained by the extremes is equal to a parallelogram, or a triangle which has an equal angle contained by the means.

PROP. XX. Similar triangles, viz. such as have equal angles, are to one another as the squares of their like sides. ABC : DEF : : CG : FH.

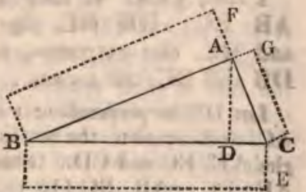
Find BK a third proportional to BC and EF the like sides, so that BC : EF : : EF : BK, and join AK, and draw KL parallel to BG. Because AB : DE : : BC : EF (18.); that is, : : EF : BK, the triangle ABK = DEF, and GK = FH (19. Cor. 3.). But GC : GK or HF : : BC : BK : : ABC : ABK = DEF.

Cor. 1. Any similar figures, viz. those composed of the same number of similar triangles similarly placed, are to one another as the squares of their like sides.

Cor. 2. If three straight lines BC, EF, BK, be proportional, the first BC is to the third BK, as any figure upon the first BC to a similar figure upon the second EF.

Cor. 3. If the area of any polygon, of which the side is 1, be multiplied by the square of any straight line, it will give the area of a similar polygon described on that line, (Prob. 12. Mens. Surfaces.).

PROP. XXI. The figure BE described upon the hypotenuse BC of a right-angled triangle ABC, is equal to the figures BF and CG, similarly described upon the other two sides BA and AC.



Draw AD perpendicular to BC. Because the angle B is common to the triangles BAC, BDA, and BAC, BDA are right angles, BD : BA : : BA : BC (18.); therefore BD : BC : : BF : BE (20. Cor. 3.). For the same reason, DC : CB : : CG : BE. Wherefore BD + DC : BC : : BF + CG : BE. Consequently, since BD + DC = BC, BF + CG = BE.

Cor. 1. If the greatest of three similar figures be equal to the sum of the other two, a right-angled triangle can be made of their like sides.

Cor. 2. The square of BC is equal to the squares of BA and AC; and, therefore, if any two of the sides be given, the third side may be found from them.

Cor. 3. If a, b, c, be three straight lines, and $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, these lines will form a right-angled triangle, of which a will be the hypotenuse.

Prop. XXII. The squares of two straight lines AB, BC, together with twice the rectangle AB × BC contained by them, is equal to the square of their sum AC.

Upon AC describe the square ADEC, and draw BG parallel to CE. Make CF = CB, and draw FHK parallel to AC. Because CF = CB, FE or DK = AB or DG; therefore DH and HC are the squares of AB and BC, and each of the figures, AH and HE, is the rectangle contained by AB and BC. But these four make the whole figure CD, which is the square of AC; therefore $AC^2 = AB^2 + BC^2 + 2\ AB \times BC$.

Cor. If AB = BC, the four figures CH, HD, AH, HE, will be squares, and equal to one another; therefore 4 times the square of AB is equal to the square of 2 AB.

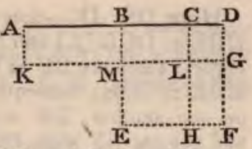
Prop. XXIII. The squares of two straight lines, AC and CB, lessened by twice the rectangle AC × CB, contained by them, are equal to the square of AB, their difference (fig. to Prop. 22.).

For CD and CH are the squares of AC and CB, and each of the figures, AF and CG, is the rectangle contained by AC and CB, and these two make AF, FG, and CH, which taken from CD + CH, leave DH the square of AB; therefore $AB^2 = AC^2 + CB^2 - 2\ AC \times CB$.

Cor. 1. The square of the sum of two straight lines exceeds the sum of their squares as much as this sum exceeds the square of their difference; and therefore 4 times the rectangle contained by two straight lines, together with the square of their difference, is equal to the square of their sum.

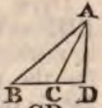
Cor. 2. The squares of the sum and difference of two lines are double of the squares of the lines.

Prop. XXIV. The rectangle contained by the sum AC, and difference DC of two straight lines, AB and BC, is equal to the difference of their squares.



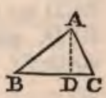
Make $BD = BA$, and upon DB make the square $DBEF$, and draw CH parallel to DF, and make $DG = DC$, and complete the rectangle $AKGD$. AC is the sum of AB and BC, and DC or DG their difference, and because $DC = DG$ or BM and $DF = AB$, the figure $ABMK = CDFH$, and $CAKL = BL + CF$; that is, to the difference of DE and EL, which are the squares of AB and BC. Therefore $AB^2 - BC^2 = (AB + BC) \times (AB - BC.)$.

Prop. XXV. The square of the side AB of a triangle opposite to an obtuse angle ACB, is greater than the squares of AC and CB, the other two sides, by twice the rectangle $BC \times CD$, contained by either side BC, and the part of it intercepted between the perpendicular AD, from the opposite angle and the obtuse angle.



For $BD^2 = BC^2 + CD^2 + 2\ BC \times CD$ (22.); add AD^2 to each, and $BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2\,BC \times CD$, but $BD^2 + DA^2 = BA^2$, and $CD^2 + DA^2 = CA^2$ (21. Cor. 2.); therefore $BA^2 = BC^2 + CA^2 + 2\,BC \times CD$.

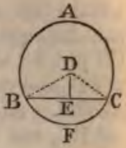
Prop. XXVI. The square of the side AC of a triangle opposite to an acute angle ABC, is less than the squares of the other two sides AB and BC, by twice the rectangle $CB \times BD$ contained by either of these sides, BC, and the part of it BD, between the perpendicular upon it from the opposite angle and the acute angle.



For $BC^2 + BD^2 = 2\,BC \times BD + DC^2$ (23.); add AD^2 to each, and $CB^2 + BD^2 + DA^2 = 2\,BC \times BD + DC^2 + DA^2$, but $BD^2 + DA^2 = BA^2$ and $CD^2 + DA^2 = CA^2$ (21. Cor. 2.); therefore $CB^2 + BA^2 = 2\ CB \times BD + CA^2$.

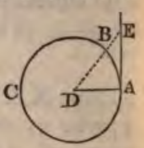
Cor. Hence the angle ABC is obtuse or acute, according as the square of AC is greater or less than the sum of the squares of AB and BC, and the difference in each case is $2\ CB \times BD$.

Prop. XXVII. A straight line, DE, drawn from the centre D, of a circle ABC, perpendicular to a chord BC, bisects the chord and the arc BFC subtended by it.



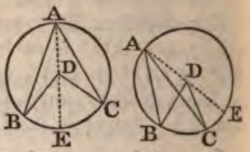
Draw DB, DC, they are equal, the angle DBE = DCE (3.) and BED, DEC, are right angles; therefore BE = EC (2.) and the angle BDE = CDE; consequently if they be laid on one another, DB will coincide with DC, and the arc BF with FC.

Prop. XXVIII. A perpendicular AE, to the diameter of a circle AC at its extremity A touches the circle.



From any point E in AE, draw ED to the centre, then DE $>$ DA or DB (4. Cor.), for the angle DAE $>$ DEA (10.), therefore E, that is, every point of AE, except A, is without the circle, and consequently AE touches it.

Prop. XXIX. An angle BDC, at the centre D of a circle, is double of the angle BAC at the circumference, when they stand upon the same arc BC.



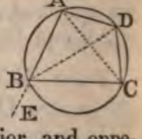
Draw ADE, then the angle BDE = DAB + DBA (10.), it is therefore = 2 BAD (3.); and for the same reason, EDC = 2 DAC; therefore, by adding or subtracting, BDC = 2 BAC.

Cor. If BDE + EDC be greater than two right angles, still the two, BDE, EDC together, are double of BAC.

Prop. XXX. Angles BAD, BED, upon the same arc BCD, or in the same segment of a circle BAED, are equal.

Join B and D with F, the centre of the circle. Then (29.) the angles BAD and BED are each of them = half the angle BFD, and consequently equal to one another.

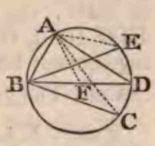
Prop. XXXI. The opposite angles ABC + ADC of a quadrilateral ABCD in a circle, are equal to two right angles.



Join AC, BD. The angle ADC = ADB + BDC = ACB + BAC (30.); therefore ADC + ABC = ACB + BAC + ABC = two right angles (10.).

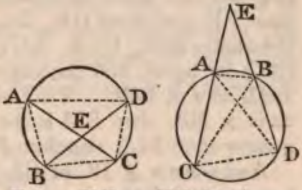
Cor. The exterior angle EBC is = the interior, and opposite angle ADC (10.).

PROP. XXXII. The angle BAD in a semicircle BAED, is a right angle, the angle BAC in a greater segment is acute, and the angle BAE in a segment less than a semicircle is obtuse.



Let F be the centre, join AF. The angle FBA=FAB, and FDA=FAD (3.); therefore BAD = ABD + ADB, and is therefore a right angle (10.). But BAC is $\angle$ BAD, and BAE $>$ BAD.

PROP. XXXIII. If through any point E, two straight lines AC, BD, be drawn, to cut the circle ABCD, and the points of their intersection with the circle be joined, the triangles thus formed are similar.

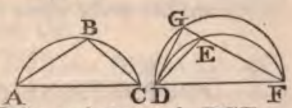


For the angle ADB=ACB (30.), and E is common; therefore the triangles ADE, CEB are similar (18.). Also ABE = ACD; therefore the triangles ABE, ECD are similar.

Cor. 1. The rectangle CE $\times$ EA = BE $\times$ ED (19.).

Cor. 2. If BE be equal to, or the same with, ED, that is, if BD be either perpendicular to the diameter AC, or touch the circle in D, then AE $\times$ EC = ED^2 (19. Cor. 1.).

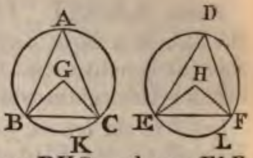
PROP. XXXIV. Segments of circles ABC, DEF, which contain equal angles ABC, DEF, and stand upon equal chords, are equal to one another.



If AC be applied to DF, and A to D, C will be on F, and the arc ABC will be on DEF; if not, let it fall on DGF, and meet FE in G, join DG; and the angle DGF = ABC = DEF, which is impossible (10.); therefore ABC coincides with DEF, and is equal to it.

Cor. The arc ABC is equal to the arc DEF.

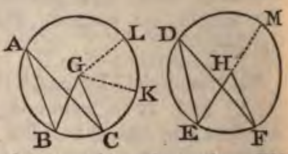
PROP. XXXV. If two equal angles, BGC, EHF, be at the centres of equal circles, ABC, DEF, the arcs BKC, ELF, upon which they stand, are equal to one another.



Draw BC, EF. Because BG GC are = EH, HF, and the angle BGC = EHF, the base BC = EF (1.); and because the angle BAC = EDF, the segment BAC = EDF (34.), therefore the remaining segment BKC = ELF, and the arc BKC = the arc ELF.

Cor. The greater angle stands upon the greater arc.

PROP. XXXVI. Angles BGC, EHF, at the centres of equal circles ABC, DEF, are to one another as the arcs BC, EF, upon which they stand. BC : EF :: BGC : EHF.



Take any number n, of arcs CK, KL, each equal to BC, so that BL $= n \times$ BC be greater than EF, and draw GK, GL, the angles BGC, CGK, KGL (35.) are equal, and the angle BGL $= n \times$ BGC. Take m such a number, that when FM $= m \times$ EF, then EM is the least multiple of EF, which is greater than BL; therefore FHM $= m \times$ EHF.

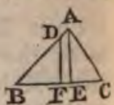
And since $n \times$ BC or BL is not less than FM or $m \times$ EF, but less than EM or $(m + 1) \times$ EF; therefore $n \times$ BGC or BGL is not less than FHM or $m \times$ EHF, but less than EHM or $(m + 1) \times$ EHF. Wherefore m is the quotient by which $n \times$ BC contains EF, and also the quotient by which $n \times$ BGC contains EHF. Therefore BC : EF :: BGC : EHF.

Cor. 1. The sector BGC = sector CGK = sector KGL; therefore BC : EF :: sector BGC : sector EHF.

Cor. 2. An angle BGC at the centre, is to four right angles as the arc BC to the whole circumference.

PROPORTIONS OF TRIGONOMETRY.

PROP. XXXVII. In any triangle ABC, the sides are to one another as the sines of their opposite angles. AB : AC :: sin. C : sin. B. (See Oblique Triangles, Rule 1.)



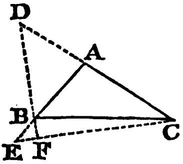
Make BD = AC, and draw AE, DF, perpendicular to BC. Making AC or BD the radius, AE is the sine of C, and DF the sine of B (Definitions of Trigonometry), and (18.) AB : BD = AC :: AE : DF :: sin. C : sin. B.

Prop. XXXVIII. Half the difference of two unequal quantities AB and BC, added to half their sum, gives the greater, and half the difference taken from half the sum, gives the less.

Make AD = BC, then AC is their sum, and BD their difference; bisect BD in E, then BE or ED is half the difference, and AE = EC half the sum, but AE + EB = AB the greater, and EC — EB = BC the less.

Cor. Half the difference BE, added to the less BC, or taken from the greater AB, gives half the sum.

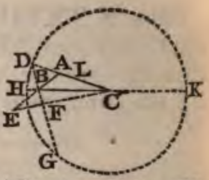
Prop. XXXIX. In any triangle ABC, of which the sides are unequal, the sum of the sides AC + AB is to their difference as the tangent of half the sum of the opposite angles B and C, to the tangent of half their difference. CA + AB : CA — AB : : tan. $\frac{1}{2}$ (B + C) : tan. $\frac{1}{2}$ (B — C).



Make AD = AB, and AE = AC, and join DB, CE, meeting one another in F. The triangles ADB, ACE, being isosceles, the angle ACE = AEC or BEF, and CDB = ABD = EBF (3.); therefore DFC = BFE a right angle, and the triangles CDF, EBF, are similar; therefore DC : EB : : DF : FB (18.); and DC = CA + AB, and BE = CA — AB; and because ABC + ACB = ACE + AEC, therefore ACF = $\frac{1}{2}$ (B + C), and BCF = $\frac{1}{2}$ (B — C); therefore AC + AB : AC — AB : : DF = tan. $\frac{1}{2}$ (B + C) : BF = tan. $\frac{1}{2}$ (B — C), the radius being CF.

Cor. Hence (by Prop. 37.) sin. BCE : sin. BEC : : BE : BC; that is, sin. $\frac{1}{2}$ (B — C) : sin. $\frac{1}{2}$ (B + C) : : AC — AB : BC. Also sin. DBC or CBF : sin. BDC : : DC : CB; that is, cos. $\frac{1}{2}$ (B—C) : cos. $\frac{1}{2}$ (B + C) : : AC + AB : BC.

Prop. XL. In any triangle ABC, four times the product of the two sides AC, AB, is to the product of the perimeter, by the excess of the sides above the base, as the square of the radius to the square of the cosine of half the angle BAC, opposite to the base.



Make $AD = AB$, and $AE = AC$, and join DB, CE, meeting in F, the angles at F are right angles. From C, with the radius CD, describe a circle meeting DF in G, and BC in H and K. Then $DF = FG$ (27.), $BK = BC + CD$, is the perimeter, and HB the excess of DC above CB. Make $AL = AB$, then $LC = BE$. Now $4\ CA \times AB = 4\ CA \times AL = CD^2 - CL^2 = CD^2 - BE^2$ (23. Cor. 1.), and $HB \times BK = DB \times BG$ (33. Cor. 1.) $= DF^2 - FB^2$ (23. Cor. 1.). But the triangles CDF, BEF, are similar; therefore $CD^2 : DF^2 :: EB^2 : BF^2$ (18.); and by alternation and division, $CD^2 - BE^2 : DF^2 - BF^2 :: CD^2 : DF^2 ::$ rad. ^{2} : cos. ^{2} CDF, (Right-angled Triangles General Rule) $= \frac{1}{2}$ BAC; therefore $4\ CA \times AB : KB \times BH = (DC + CB) \times (DC - CB) ::$ rad. ^{2} : cos. $^2\ \frac{1}{2}$ BAC.

Cor. Since $HB = KC - CB = KB - 2\ BC$, and if $P = \frac{1}{2}$ BK, then $CA \times AB : P \times (P - BC)$: rad. ^{2} : cos. $^2\ \frac{1}{2}$ A.

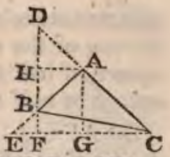
Prop. XLI. In any triangle ABC, if AD be perpendicular to BC, the rectangle or product of the sum, and difference of the sides AC, AB, is equal to the product of the base BC, by the difference between it and the double of one of its segments.

From A, with the greater side AC for a radius, describe a circle meeting AB produced in E and F, and CB in G; then $BE = CA + AB$, and $BF = CA - AB$, and because $CG = 2\ CD$ (27.), $GB = 2\ CD - CB$, but (33. Cor.) $CB \times BG = EB \times BF$.

Cor. If $EB \times BF \div CB = R$, then $CD = \frac{1}{2}(BC + R)$, and $BD = \frac{1}{2}(BC - R)$.

MENSURATION.

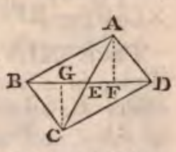
Prob. XLII. Any triangle ABC, is a mean proportional between the rectangle contained by half the perimeter and its excess above the base, and the rectangle contained by half the sum and half the difference of the base BC, and the difference of the sides AC and AB. (Mens. Surfaces, Prob. 6.).



Make $AD = AB$, and $AE = AC$, and join DB and CE, meeting one another in F, and parallel to them draw AG and AH. The angles at F, G, and H, are right angles, as in Prop. 39. And $DH = HB$; $CG = GE$; $HF = \frac{1}{2}(DF + FB)$, and $FG = \frac{1}{2}(CF - FE)$. The rectangle $\frac{1}{2}(DC + CB) \times \frac{1}{2}(DC - CB) = \frac{1}{2}(DF + FB) \times \frac{1}{2}(DF - FB)$ (41.) $= FH \times HD = AG \times HD$. And for the same reason, $\frac{1}{2}(CB + BE) \times \frac{1}{2}(CB - BE) = CG \times GF$. And because the triangle $ACE = CG \times GA$, or $CG \times FH$, and the triangle $CBE = CG \times BF$ (Prob. 4. Mens. Surfaces), therefore the triangle $ABC = CG \times BH$, or $CG \times DH$. And the triangles AGC, DHA, are similar; therefore $AG : GC :: DH : HA = FG$, and multiplying the two first by DH, and the two last by GC, the rectangle $AG \times DH$, or $DH \times HF : GC \times DH :: DH \times GC : FG \times GC$; that is, $\frac{1}{2}(DC + CB) \times \frac{1}{2}(DC - CB)$: the triangle ABC :: as the triangle $ABC : \frac{1}{2}(CB + BE) \times \frac{1}{2}(CB - BE)$.

Cor. If P be $\frac{1}{2}$ the perimeter, then the triangle $ABC = \sqrt{((P \times (P - BC) \times (P - AC) \times (P - AB))}$, for $\frac{1}{2}(BC + BE) = \frac{1}{2}(BC + CA - AB) = P - AB$, and $\frac{1}{2}(BC - BE) = \frac{1}{2}(BC + AB - AC) = P - AC$.

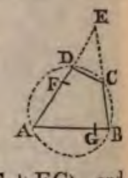
Prop. XLIII. In any quadrilateral ABCD, one-fourth of the excess of the squares of two opposite sides, AB and CD, above the squares of the other two, AD and BC, is to the area, as radius to the tangent of the angle formed by the diagonals. (Mens. Prob. 10, Note 2.).



Draw AF, CG, perpendicular to the diagonal BD. Because $EF = AE \times c$ (putting c for the cosine of the angle at E), and $GE = CE \times c$; therefore $GF = AC \times c$. And because $AB^2 - AD^2 = BF^2 - FD^2$ (41.) $= BG^2 + GF^2 + 2BG \times GF - FD^2$, and $DC^2 - CB^2 = DG^2 - GB^2 = DF^2 + FG^2 + 2\,DF \times FG - BG^2$ (22.); therefore $AB^2 + DC^2 - AD^2 - CB^2 = 2\,FG^2 + 2\,FG \times (BG + DF) = 2\,FG \times (BG + GF + FD) = 2\,FG \times BD = 2\,BD \times AC \times c$; and the area $= \frac{1}{2}BD \times AC \times s$. ($s$ = sine AED) (Prop. 5, Mens. Surfaces); therefore $\frac{1}{4}(AB^2 + DC^2 - BC^2 - AD^2)$: the area :: $c : s$ :: rad. : tan. AED. *Solution of right-angled triangles.*

Prop. XLIV. If a quadrilateral ABCD be inscribed in a circle, the area of the figure is a mean proportional between the excess of the square of half the sum of two

adjacent sides AD, DC, above the square of half the difference of the other two, AB, BC, and the excess of the square of half the sum of the latter AB, BC, above the square of half the difference of the former AD, DC.



Let $AF = \frac{1}{2}(AD + DC)$, and $AG = \frac{1}{2}(AB + BC)$, then $DF = \frac{1}{2}(AD - DC)$, and $GB = \frac{1}{2}(AB - BC)$; and $AF^2 - BG^2$: area :: area : $AG^2 - DF^2$. Produce AD, BC to E. Because the triangles ABE, DCE, are similar, $AB : DC :: AE : EC :: BE : ED$; therefore (putting $P = \frac{1}{2}$ sum of AB, BE, and AE), $AB : DC :: \frac{1}{2}(AB + AE + EB) = P : \frac{1}{2}(DC + DE + EC)$, and (*by conv.*) $AB : BA - DC :: P : \frac{1}{2}(AB + AD + BC - CD) = AG + DF$. Again, $AB : CD :: \frac{1}{2}(AB + BE - AE) = P - AE : \frac{1}{2}(CD + DE - EC)$, and (*comp.*) $AB : AB + CD :: P - AE : \frac{1}{2}(AB + BC + CD - AD) = AG - DF$, and multiplying the corresponding terms of these proportions $AB^2 : AB^2 - CD^2 :: P \times (P - AE) : AG^2 - DF^2$. In like manner it may be proved that $AB^2 : AB^2 - CD^2 :: (P - BE) \times (P - AB) : AF^2 - BG^2$. But because the triangles ABE, DCE are similar $AB^2 : DC^2 :: ABE : DCE$ and $AB^2 : AB^2 - DC^2 :: ABE : ABCD$. Therefore $P \times (P - AE) : AG^2 - DF^2 :: ABE : ABCD$, and (*altern.*) $AG^2 - BF^2 : ABCD :: P \times (P - AE) : ABE$; that is, (Prop. 42.) $:: ABE : (P - BE) \times (P - AB)$, or $:: ABCD : AF^2 - BG^2$. Therefore ABCD is a mean proportional between $AG^2 - BF^2$, and $AF^2 - BG^2$.

Cor. Hence the quadrilateral $ABCD = \sqrt{((AF^2 - BG^2) \times (AG^2 - BF^2))}$.

Prop. XLV. A quadrilateral ABCD, inscribed in a circle, is a mean proportional between the rectangle under the excesses of half the perimeter above two of its sides, and the rectangle under its excesses above the other two sides. (Mens. Surfaces, Prob. 10, Note 3.)

Let P be half the perimeter, then $AF^2 - BG^2 = (AF + BG) \times (AF - BG)$ (Prop. 13.) $= \frac{1}{2}(AD + DC + AB - BC) \times \frac{1}{2}(AD + DC - AB + BC) = (P - BC) \times (P - AB)$, and $AG^2 - DF^2 = \frac{1}{2}(AB + BC + AD - DC) \times \frac{1}{2}(AB + BC - AD + DC) = (P - DC) \times (P - AD)$; therefore ABCD is a mean proportional between $(P - BC) \times (P - AB)$, and $(P - DC) \times (P - AD)$.

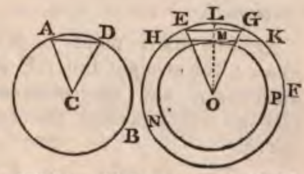
Prop. XLVI. The area of any circle ABD is equal to the rectangle contained by the radius AC, and a

straight line equal to half the circumference ABD. (Mens. Surfaces, Prob. 18.).

If not, let the rectangle be less than the circle ABD, or equal to the circle EGM. Draw FD, touching this circle in E, and meeting the circumference ABD in F and D, and join CD, meeting the arc EG in H. Let EG be a fourth part of the circumference EGM. From EG take away its half, and from the remainder its half, and so on, till the arc EK is found less than EH. Draw CKL, and make EN = EL. Then LN is the side of a regular polygon, described about the circle EGML; and it is plain that this polygon is less than the circle ABD. Because the triangle $NLC = \frac{1}{2} NL \times CE$, the polygon is $= \frac{1}{2}$ the perimeter $\times$ CE. But the perimeter is less than the circumference ABD, and CE is less than CA; therefore the polygon is less than $\frac{1}{2}$ the circumference $ABD \times CA$; that is, less than the circle EGM, which it contains; therefore the rectangle is not less than the circle ABD. And it may be shown, by a similar construction about ABD, that it is not greater. Therefore the circle is equal to the rectangle contained by the radius and the half of the circumference.

Cor. Any sector of a circle is equal to the rectangle or product of the radius, and half the arc of the sector.

PROP. XLVII. The circumferences of the circles ABD, EFG, are to one another as their radii. (Mens. Surfaces, Prob. 13.).



If possible, let the radius AC, be to the radius EO, as the circumference ABD to a circumference MNP, less than EFG. Draw the radius OML, and HMK touching the circle MNP in M; and let LF be a fourth part of the circumference EFG. Take away its half, and the half of the remainder, and so on, till an arc LG is found less than LK, and draw GE parallel to HK, it will be the side of a regular polygon in the circle EFG; and this polygon is greater than MNP. Let AD be the side of a similar polygon inscribed in the circle ADB, and join EO, OG, AC, CD. The triangles ACD, EOG, being similar AC : EO : : AD : EG; that is, as the perimeter of the polygon in ADB to the perimeter of the polygon in EFG; but AC : EO : : circumference ADB : circumference MNP; the perimeters,

therefore, are as these circumferences; but this is impossible, for the perimeter of the polygon in ADB is less than the circumference; and, on the contrary, the perimeter of the polygon in EFG is greater than the circumference MNP. Therefore AC is not to EO as the circumference ADB to a circumference less than EFG; and in the same manner it may be shown that EO is not to AC as the circumference EFG, to a circumference less than ADB. Therefore AC : EO : : the circumference ABD : the circumference EFG.

Cor. 1. Hence circles are to one another as the squares of their radii, or of their diameters.

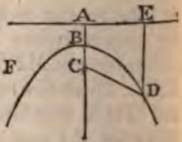
Cor. 2. If p be the circumference of a circle, of which the diameter is 1, or $\frac{1}{2}$ the circumference, of which the radius is 1, then $1 : p :: CA : \frac{1}{2}$ the circumference $ADB = p \times CA$, and therefore $p \times CA \times CA = p \times CA^2 =$ area of the circle ADB. (Mens. Surfaces, Prob. 19.).

Section II.

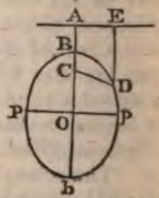
PROPERTIES OF CONIC SECTIONS.

DEFINITIONS.

1. If a point D move in a plane, and its distances from a fixed point C, and from a straight line AE, both in that plane, have always the same ratio to one another, the moving point will describe a *curve*, called a *line* of the *second order*, or a *conic section*.



2. The fixed point C is called the *focus;* the straight line AE is called the *directrix;* and the constant ratio of CD to DE is called the *ratio* of the curve.

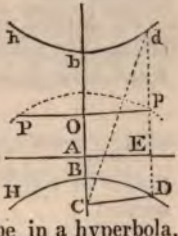


3. The straight line CA, drawn through the focus C, perpendicular to AE, is called the *axis,* or the *transverse axis,* and the point B, in which it cuts the curve, is called the principal vertex.

Cor. Hence CB : BA : : CD : DE, or in the ratio of the curve.

4. If CB be equal to BA, or the ratio of the curve be that of equality, the curve is called a *parabola,* as DBF.

5. If CB be less than BA, or the ratio be one of minority, the curve is called an *ellipse,* as DBP.



Cor. If AC be produced beyond C to b, so that $Ab : bC :: AB : BC$, the point b will be in the ellipse, which, therefore, contains a space.

6. If CB be greater than BA, or the ratio be one of majority, the curve is called a *hyperbola,* as DBH.

Cor. If CA be produced beyond A, so that $Cb : bA :: CB : BA$, the point b will be in a hyperbola, similar, and equal to DBH, and described in the same way; it is called the *opposite hyperbola.*

7. The straight line Bb in the ellipse and hyperbola is properly the axis, B and b its vertices, and the point O in which it is bisected is called the *centre.*

8. A straight line Pp, drawn through the centre O, perpendicular to the transverse axis, is called the *conjugate* axis, and the points P, p in the ellipse in which it meets the curve, are called its vertices. But in the hyperbola, the vertices P, p are the points in which it meets the circle described from B, with the radius OC.

9. Every straight line which is perpendicular to the directrix of a parabola, or which passes through the centre of an ellipse or a hyperbola, is called a *diameter;* and the point in which it meets the curve is its vertex.

10. A straight line which meets the curve, and does not cut it, is called a *tangent;* and if the straight line from the point of contact to the focus be parallel to the directrix, the tangent is called the *focal tangent.*

11. A straight line parallel to a tangent, is said to be *ordinately applied* to the diameter which passes through the point of contact, and the part of it between the curve and that diameter is called an ordinate.

12. The segments of a diameter intercepted between an ordinate and its vertices, are called *abscissas* to that ordinate.

13. Straight lines drawn through the centre of a hyperbola parallel to the straight lines which join the vertices of the axes, are called *asymptotes.*

14. Two diameters of the ellipse or hyperbola, each of which is parallel to the tangent in the vertex of the other, are called *conjugate diameters.*

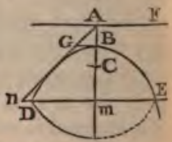
15. Four times the segment of a diameter of the parabola

between its vertex and the directrix, is called the *parameter* of that diameter.

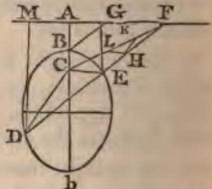
16. A third proportional to two conjugate diameters of the ellipse or hyperbola, is called the *parameter* of that diameter, which is the first of the three proportionals.

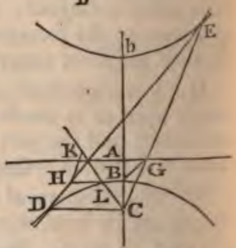
PROPOSITIONS.

PROP. XLVIII. Problem. To find the point or the points in which a given straight line, DE, meets a conic section.



If DE be parallel to the directrix AF, draw BG parallel to AF, and make BG = BC, and join AG, and let it meet DE in *n*, and let the axis meet DE in *m*. From C, with the radius *mn*, describe a circle meeting DE, in the points D and E. Because the perpendicular from D upon AF is equal to A*m*, CD : perpendicular : : *nm* : *m* A : : GB = CB : BA, therefore the point D is in the curve, and for the same reason, E is in the curve.





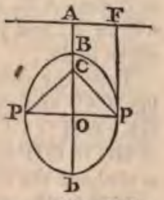
If DE meet the directrix in F, join FC. If DE be parallel to AC, make FH = AB. If not, draw BG parallel to DE, and make FH = BG. Then with BC for a radius, from H, cut CF in K and L, and through C draw CD and CE, parallels to HK and HL, the points D and E are in the curve. Draw DM perpendicular to the directrix. The triangles DMF, BAG are similar; therefore MD : DF : : AB : BG, and DF : DC : : FH : HK : : GB : BC; hence MD : DC : : AB : BC. Therefore D is in the curve, and for the same reason E is in the curve.

In the parabola AB = BC; and therefore, if DE be perpendicular to the directrix, the point K will fall on F; in which case the straight line DE will meet the curve only in one point D.

In the hyperbola, where AB is less than BC, the point K may fall above F; in which case DE meets each of the opposite hyperbolas in one point.

In the ellipse in which CB is less than BA, the circle described from H may not meet CF; in which case DE will not meet the curve. And a straight line may be drawn between the opposite hyperbolas, so as not to meet either of them, but two other hyperbolas which have the conjugate diameter of the former for their transverse, and the transverse for their conjugate.

PROP. XLIX. Problem. Given the directrix, the focus, and the ratio of an ellipse, or of an hyperbola, to find the axes.



Having drawn AC from the focus C perpendicular to the directrix, make the sum of the terms of the ratio to the first term, as AC to CB, and their difference to the first, as AC to Cb. Then B and b are the extremities of the transverse axis. And because AB : BC :: Ab : bC; therefore AB : BC :: $\frac{1}{2}$(AB + Ab) : $\frac{1}{2}$(BC + bC), or :: $\frac{1}{2}$(Ab — AB) : $\frac{1}{2}$(bC — BC), that is AB : BC :: AO : OB, or :: OB : OC; therefore OB and OC are given.

Join Cp in the ellipse, and Bp in the hyperbola. Then pF = AO : Cp :: AO : OB, therefore Cp = OB, and in the hyperbola Bp = OC; hence in both curves Op^2 is the difference between OB^2 and OC^2, it is therefore = BC × Cb, or = AC × CO.

Suppose AC to be 14, and the ratio of the ellipse be that of 3 to 4, or of the hyperbola that of 4 to 3. In the ellipse 7 : 3 :: 14 : 6 = CB, and AB = 8. Also 1 : 3 :: 14 : 42 = Cb; therefore Bb = 48, OB = 24, OC = 18, OA = 32, and Op = $\sqrt{OB^2 - OC^2}$ = 15·8745.

In the hyperbola 7 : 4 :: 14 : 8 = CB and AB = 6. Also 1 : 4 :: 14 : 56 = Cb; therefore Bb = 48, OB = 24, OC = 32, and Op = $\sqrt{OC^2 - OB^2}$ = 21·166.

Cor. 1. Hence OB^2 = AO × CO, and OP^2 = AC × CO = BC × Cb.

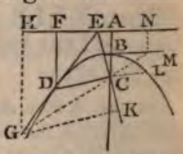
Cor. 2. Hence AC : CB :: Cb : CO, and AC : AB :: Cb : BO.

Cor. 3. If the axes of an ellipse or an hyperbola be given, the focus, the directrix, and the ratio of the curve may be found. Let the transverse axis be 80, and the conjugate 60. In the ellipse OC = $\sqrt{OB^2 - OP^2} = \sqrt{40^2 - 30^2} = 10\sqrt{7}$ = 26·4575, OA = OB^2 ÷ OC = 60·4743, AB = 20·4743, and

$BC = 13{\cdot}5425$, and the ratio of the curve that of $10\sqrt{7} : 40$, or of $\sqrt{7} : 4$, or of $2{\cdot}64575 : 4$.

In the hyperbola $OC = BP = \sqrt{40^2 + 30^2} = 50$, $OA = 32$, and the ratio of the curve that of $OC : OB$, or of 5 to 4.

Prop. L. Problem. Given the directrix, and the focus of a conic section, to draw a straight line which shall touch the curve at a given point.



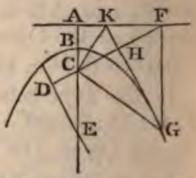
Let D be the given point in the curve. Draw DC to the focus, and perpendicular to it draw CE, and let it meet the directrix in E, and join DE, it will touch the curve in D.

Take any point G in DE, and draw GK parallel to DC, and join GC, and draw DF, GH perpendicular to the directrix. $GK : GH :: CD : DF$, but $GC > KG$; therefore the ratio of GC to GH is greater than the ratio of the curve, and so G is without the curve, and DE touches it.

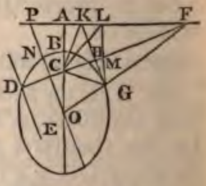
If B be the given point, so that CL perpendicular to CB, is parallel to AF, then BM parallel to AE touches the curve, for $CM > CB$ and $MN = BA$; therefore $CM : MN > CB : BA$.

Cor. If CL be parallel to AF, then AL touches the section, and is the focal tangent.

Prop. LI. Problem. Given the directrix AF, and the focus C of a conic section, to draw a straight line which shall touch the curve, and be parallel to a given straight line DE.



From the focus C, draw CD perpendicular to DE, and let it meet the directrix in F, draw the diameter FG, and through its vertex G draw GH parallel to DE, and it will touch the curve at G. Join GC and CK.



In the parabola. Since the angles at H are right angles. $HCG + CGH$ = a right angle = GFK of which GCH = GFH, because $GC = GF$; therefore $CGK = CFK$, and the four points C, G, F, K, are in the circumference of a circle, of which GK is the diameter; therefore (31.) GCK is a right angle, and GK touches the curve (49.)

In the ellipse and hyperbola. Draw GL perpendicular to

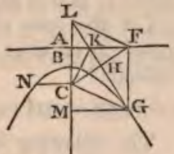
the directrix, and let it meet CF in M, then $GM : GL :: OC : OA :: OC^2 : OB^2$ (48.), or $:: CG^2 : GL^2$; therefore the triangles CGM, CGL are similar, and the angle $GCM = GLC$; hence $CGH = CLK$, and the four points C, G, L, K are in the circumference of a circle, of which GK is the diameter; therefore GCK is a right angle and GK a tangent.

Cor. 1. A straight line FC, drawn to the focus from the intersection F of a diameter, with the directrix, is perpendicular to the ordinates to that diameter.

Cor. 2. Tangents at the vertices of the same diameter are parallel to one another.

Cor. 3. Two diameters OG, ON, one of which ON is parallel to the tangent GK in the vertex of the other, are conjugate. Because in the triangle OFP, FC and OC are perpendicular to the sides OP and PF, therefore, since the perpendiculars from the angles of a triangle upon the opposite sides all pass through the same point, PC will be perpendicular to OF; that is, OF is parallel to the tangent in N.

Prop. LII. Problem. Given the axis, the directrix, and the focus, to find the point in which a tangent GK meets the axis.



Through G draw the diameter GF, meeting the directrix in F. Join GC, CK and CF, and draw FL parallel to CG, and GM parallel to AF. CF is perpendicular to the tangent GK, and CK to CG. And because in the triangle LFC, FK and CK are perpendicular to the sides LC and FL; consequently LK is perpendicular to the third side CF, which is therefore in the same straight line with KG; that is, GK meets the axis in L.

To find the point L. *In the parabola,* $LC = FG = AM$, and $AB = BC$; therefore $LB = BM$, and $LC = CG$.

In the ellipse and hyperbola. $OL : OC :: OF : OG :: OA : OM$; therefore $LO \times OM = OA \times OC = OB^2$, and $OB^2 \div OM = OL$.

Cor. 1. If the tangent meet the conjugate axis of the ellipse or hyperbola in N, and GR be parallel to BO, the rectangle $NO \times OR = OP^2$. For the triangle AFC is similar to LON, and OAF to OMG; hence $LO : ON :: FA : AC$ and $MO : MG :: OA : AF$; therefore $LO \times OM : NO \times MG = NO \times OR :: OA : AC$ or $:: OB^2 : OP^2$, and $LO \times OM = OB^2$; therefore $NO \times OR = OP^2$.

Cor. 2. The rectangle $OM \times ML = BM \times Mb$.

For $LO \times OM = OB^2$ take OM^2 from each, and $OM \times ML = MB \times Mb$.

PROP. LIII. Problem. Given the abscissa and the parameter of a parabola, to find GM the ordinate to the axis.

The triangles LMG, FAC are similar, (see last figure); hence $LM : MG :: AF = MG : AC$; therefore $MG^2 = LM \times AC = BM \times 2\,AC = BM \times$ parameter.

Let AB be 10, and the abscissa BM $22\frac{1}{2}$, the parameter is 40, and $40 \times 22\frac{1}{2} = 900 = MG^2$; therefore MG is 30.

PROP. LIV. Problem. Given the two axes of an ellipse, or of a hyperbola, and the abscissa, to find the ordinate GM.

Because the triangles FAC, LCH are similar, (see last figure,) the angle $AFC = CLH$, and therefore the triangle FAC is similar to LGM, and $LM : MG :: FA : AC$, and $OM : MG :: OA : AF$; hence $LM \times OM : MG^2 :: OA : AC :: OB^2 : OP^2$, and $LM \times OM = BM \times Mb$; therefore $BO^2 : OP^2 :: BM \times Mb : MG^2$.

Let the axes of an ellipse be 210 and 150, and the abscissa cut off from the vertex of the first be 42. What is the ordinate? Ans. $(210 - 42) \times 42 = 7056$.

$$210 : 150 :: \sqrt{7056} = 84 : 60 \text{ the ordinate.}$$

The following formulæ exhibit the rules for finding any of the quantities concerned.

Let the ratio of the curve be that of 1 to n, or in the parabola of n to n, AC the distance of the focus from the directrix $= d$, the abscissa $BM = x$, the ordinate $MG = y$, the subtangent $ML = t$, and in the ellipse and hyperbola, let OB the semi-transverse axis be $= a$, OP the semi-conjugate $= b$, OC the distance from the focus to the centre $= c$, and the parameter $= p$.

In the parabola. 1. $AB = BC = \frac{1}{2}d$. 2. $AM = \frac{1}{2}d + x = CG$. 3. $LM = 2\,x$. 4. $MG = \sqrt{(x + \frac{1}{2}d)^2 - (x - \frac{1}{2}d)^2} = \sqrt{2\,dx} = \sqrt{px} = y$. 5. $LG = \sqrt{2\,x \times (d + 2\,x)}$. 6. $CH = \frac{1}{2}\sqrt{(2x + d) \times d} = \frac{1}{2}\sqrt{y^2 + d^2}$. 7. $CN = \sqrt{2d \times \frac{1}{2}d} = d$.

In the ellipse and hyperbola. 1. $BC = \frac{d}{1+n} = a \times \frac{1-n}{n}$. 2. $AB = \frac{nd}{1+n} = a \times (1 - n)$ or $a \times (n - 1)$. 3. $BO = a =$

$\frac{nd}{1-n^2}=\frac{nd}{r^2}$ (putting $r^2=1-n^2$ in the hyperbola, or $=n^2-1$ in the ellipse). 4. $CO=\frac{d}{r^2}=\frac{a}{n}=\sqrt{a^2-b^2}$ in the ellipse, and $=\sqrt{a^2+b^2}$ in the hyperbola. 5. $OP=\frac{d}{r}=\frac{ar}{n}=b$. 6. $OA=\frac{n^2d}{r^2}=\frac{a^2}{\sqrt{a^2\mp b^2}}$. 7. $OM=a\mp x=\frac{nd}{r^2}\mp x$. 8. $OL=\frac{a^2}{a\mp x}=\frac{n^2}{r^2}\times\frac{d^2}{nd\mp r^2x}$. 9. $ML=\frac{2ndx\mp r^2x^2}{nd\mp r^2x}=\frac{2ax\mp x^2}{a\mp x}$. 10. $OM\times ML=BM\times Mb=\frac{2ndx}{r^2}\mp x^2=(2a-x)\times x$. 11. MG the ordinate $=\frac{\sqrt{2ndx\mp r^2x^2}}{n}-\frac{b}{a}\times\sqrt{2ax\mp x^2}=\frac{r}{n}\times\sqrt{2ax\mp x^2}$.

EXAMPLES.

1. In the parabola is given the parameter $p=4$ to find the distance of the focus from the directrix, and from the principal vertex. Ans. $AC=d=\frac{1}{2}p=2$, and $BC=\frac{1}{4}p=1$.

2. In the parabola are given the distance of the focus from the directrix $d=2$ and absciss $BM=x=9$, to find the distance of the ordinate from the directrix, and from the tangent at the extremity of the ordinate LM.

Ans. $AM=\frac{1}{2}d+x=1+9=10$, $LM=2x=18$. Hence $CM=x-\frac{1}{2}d=8$.

3. In the parabola are given the distance of the focus from the directrix $=2$, or the parameter and the absciss 9, to find the ordinate MG. Here $MG=\sqrt{(x+\frac{1}{2}d+x-\frac{1}{2}d)\times(x+\frac{1}{2}d-x+\frac{1}{2}d)}=\sqrt{2dx}=\sqrt{px}=\sqrt{4\times 9}=6$.

Again, let the parameter be 9 and the abscissa 16, then $\sqrt{9\times 16}=\sqrt{144}=12$ the ordinate.

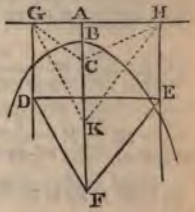
Again, let $p=54$ and $x=6$, the ordinate is $\sqrt{6\times 54}=18$.

4. Given the ordinate $y=16$ and the parameter $p=8$, to find the abscissa. Ans. $16^2\div 8=32$.

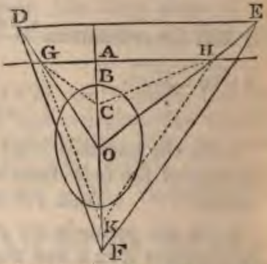
5. In the ellipse are given the ratio of the curve $1:n=3$ and the distance of the focus from the directrix $d=12$, to find their distances from the principal vertices B and b. $BC=d\div n+1=12\div 4=3$; $AB=3\times 3=9$; $Cb=d\div n-1=12\div 2=6$, and $Ab=6\times 3=18$.

Prop. LV. Theorem. If two sides DE, EF of a triangle DEF be ordinately applied to the diameters

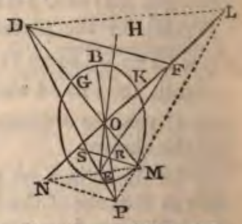
AF, DG, of a conic section, which pass through their opposite angles F and D, the third side DF shall also be ordinately applied to the diameter EH, which passes through its opposite angle E.



First, Let one of the diameters AF be the axis, and let the diameters meet the directrix in A, G, and H. Draw GC, HC to the focus, and draw GK perpendicular to CH, meeting AF in K, and join HK. Because GK is perpendicular to CH, or CH to GK, and CA to GH; consequently GC is perpendicular to KH, that is, HK is an ordinate to GD, and is therefore parallel to EF. Hence in the parabola KF = EH or = DG; and therefore DF is parallel to GK, which is an ordinate EH. In the ellipse and hyperbola OK : OF : : OH : OE or : : OG : OD; therefore DF is parallel to GK, an ordinate to EH.

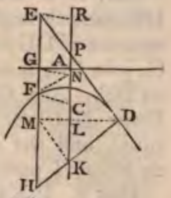


Next, Let none of the diameters be the axis. Let DE and EF meet the axis in P and R. Draw DL and MEN perpendicular to the axis, and let FK meet them in L and N, and let DG meet MN in M. Join PL, PN, and MR, and let MR meet FK in S. Because EN, EP are ordinates to the diameters PR, KN, therefore PN is an ordinate to EH. For the same reason MR is an ordinate to EH, and PL an ordinate to GD; therefore PL is parallel to EF and PN to MS. Wherefore *in the parabola*, SN = PR = LF and SF = LN = DM, and DF is therefore parallel to MS. And *in the other curves* OS : ON : : OR : OP, that is : : OF : OL, and alternately OS : OF : : ON : OL, that is, : : OM : OD; therefore DF is parallel to SM, and is an ordinate to the diameter EH.



PROP. LVI. Theorem. If a tangent to a conic section DE meet a diameter EF, and from the point of

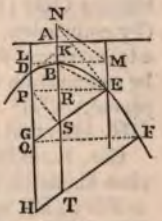
contact D, an ordinate DH be applied to that diameter. Then, in the parabola, the segment of the diameter EH between the tangent and the ordinate is bisected in the vertex F. And in the ellipse and hyperbola, the semi-diameter OF is a mean proportional between the segments of it OE and OH from the centre, intercepted by the tangent and the ordinate.



Let the axis meet the tangent in P, and the ordinate in K. Draw DM parallel to the directrix, and FN to touch the curve in F, and let the axis meet them in L and N. Join FC, GN they are parallel (50.) and draw ER parallel to FC. Because DM and DK are ordinates to the diameters through K and M, therefore MK is an ordinate to the diameter through D, and it is therefore parallel to DE. Wherefore *in the parabola*, EM = PK, and GM = AL = PC; therefore CK = EG = RN, and NK = CR, that is, HF = FE.

In the ellipse and hyperbola, OK : OP :: OM : OE, but because OB is a mean proportional between OL and OP, and also between OC and OA; therefore OP : OC :: OA : OL :: OG : OM. Wherefore, (*by inverse equality*) OK : OC :: OG : OE :: ON : OR, and *alternately* OK : ON :: OC : OR, that is, OH : OF :: OF : OE.

Prop. LVII. Theorem. If from two points E and F of a parabola ordinates EG, FH be applied to any diameter DH. The squares of the ordinates will be to one another as the abscissas DG and DH between them and the vertex.



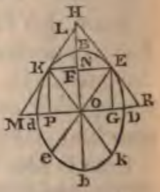
Draw LBM, EP, FQ, parallel to the directrix, and draw the tangents DK and EN, and join BE and NM. Because ER is an ordinate to NB, therefore NB = BR = EM; therefore NM is parallel to BE, and the triangle NBE = MBE, and the whole triangle NRE = BREM. In the same manner it may be proved, that BLPR = KDPR. And because PE, ES are ordinates to the diameters through S and P, PS is an ordinate to the diameter through E, and is parallel to EN, therefore NR : RS :: ER :

RP, and NR : RS : : triangle NRE : RES, and ER : RP : : parallelogram RM : RL, and the triangle NRE = RM, therefore the triangle RSE = RL = KDPR, and by adding PRSG, the triangle EPG = KDGS. In the same manner it may be proved, that the triangle FQH = parallelogram KDHT. And the triangles are similar; therefore GE^2 : FH^2 : : EPG : FQH : : KG : KH : : DG : DH.

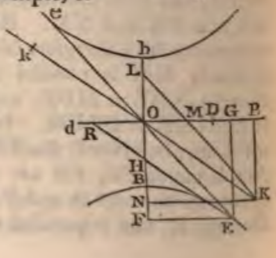
Cor. 1. If the ordinate EF to the diameter DG, pass through the focus C, EF is $\frac{1}{2}$ the parameter of DG. Let DG meet the directrix in G, join GC, it is perpendicular to EF, and DG = DF. Also GE will touch the curve at E. Draw EH parallel to DG, the triangles GCE, GHE, are equal, and the angle GEC = GEH = FGE; therefore FE = FG = $\frac{1}{2}$ parameter.

Cor. 2. If KL be another ordinate to DG, LK^2 = DL $\times$ parameter. For EF^2 : LK^2 : : DF : DL : : DF $\times$ par. : to DL $\times$ par. and EF^2 = DF $\times$ 2 FG = DF $\times$ par.; therefore KL^2 = DL $\times$ parameter.

Prop. LVIII. Theorem. If from any point E of the ellipse or hyperbola, an ordinate be applied to any diameter B*b*. The square of the diameter D*d*, which is parallel to the ordinate, is to the square of the ordinate EF, as the square of the diameter B*b*, to which the ordinate is applied, to the difference between the square of this semidiameter and the square of the segment of it between the centre O and the ordinate EF.



Let the tangent at E meet the diameters B*b*, and D*d* in H and R, and draw EG parallel to B*b*, it is an ordinate to D*d*. Therefore OB^2 = FO $\times$ OH, and OD^2 = OG $\times$ OR. Also OD^2 : OG^2 : : OR : OG = EF, that is, : : OH : HF. But because OB^2 : OF^2 : : OH : OF, therefore, (by conversion, when B*b* is in the ellipse, or a transverse of the hyperbola, and by composition when OB is a conjugate,) OB^2 : $OB^2 \mp OF^2$: : OH : HF, that is, : : OD^2 : EF^2.



Cor. 1. When B*b* is a transverse diameter, the rectangle HF $\times$ FO = BF $\times$ F*b* is = OB^2 — OF^2 (50.); therefore Bb^2 : Dd^2 : : BF $\times$ F*b* : EF^2.

Cor. 2. The squares of ordinates to the same diameter are to one another as the rectangles contained by the abscissas between them and the vertices.

Prop. LIX. Theorem. If from the vertices E and K of two conjugate diameters of the ellipse or hyperbola ordinates EF, and KN be applied to any other diameter Bb. The rectangle $BF \times Fb$ contained by the abscissas of that diameter between one of the ordinates and its vertices is equal to the square of ON the segment between the other ordinate and the centre.

Let the tangents at E and K meet the diameter Bb in H and L. Because HE is parallel to OK, KL to OE, and KN to EF, the triangles KON, FEH are similar, and likewise OKL, HEO; therefore $FH : HE :: NO : OK$, and $HE : HO :: KO : OL$, and, *by equality*, $FH : HO :: NO : OL$, and multiplying the two first by OF, and the other two by ON, the rectangle $HF \times FO : HO \times OF :: ON^2 : LO \times ON$, but $HO \times OF = OB^2 = LO \times ON$, therefore $ON^2 = HF \times FO = BF \times Fb$. And in the same way we prove that $OF^2 = BN \times Nb$.

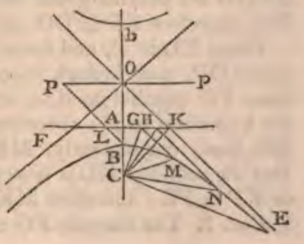
Cor. 1. $Bb : Dd :: ON : EF$, and $:: OF : KN = OP$.

Cor. 2. In the ellipse $OF^2 + ON^2 = OB^2$, but in the hyperbola $OF^2 - ON^2 = OB^2$.

Cor. 3. If KP be parallel to Bb, then FP is parallel to BD or Bd.

Prop. LX. Theorem. The asymptotes and the hyperbola continually approach, and at length come nearer to one another than by any given distance, but they never meet.

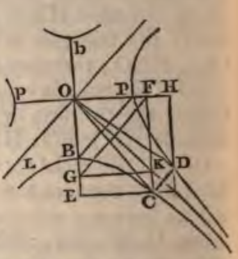
Join BP, Bp the vertices of the axes, and parallel to them draw OE, OF, these are the asymptotes. Let G be any point in the directrix, and draw GM parallel to the asymptote, and join GC, and make the angle $GCM = CGM$, therefore M is in the hyperbola. Let GK be any given distance, and take KH less than KG, and draw HN parallel to the asymptote. Join HC, and make the angle $HCN = CHN$, then N is in the hyperbola, and it is nearer to the asymptote than M, and it is also farther from B, for the



angle HCN is greater than GCM, because CHK > CGK, and KHN = KGM. If the hyperbola meet the asymptote in E. Join EC and CK, then ECK = EKC = a right angle, which is impossible, therefore they never meet.

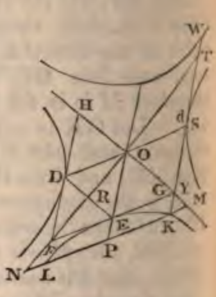
That CK is perpendicular to the asymptote, may be proved thus: The triangles OPB, OAK are similar; hence KO : OA : : PB = OC : OB : : OB : OA; therefore OK = OB, and the angle OKC = OAK = a right angle.

Prop. LXI. Theorem. The straight line CD, which joins the vertices of two conjugate diameters OC, OD is parallel to OL, one of the asymptotes, and is bisected by the other OK.



Draw CE, CF, DG, DH, parallel to the axes OB, OP, and join BP, FG. They are parallel to one another, and BP is bisected by the asymptote OK; therefore FC, GD will meet one another in OK, let it be at K. Then (58, Cor. 1.), OE : OG : : OH : OF, that is, FC : FK : : DG : GK; therefore CD is parallel to FG or BP, and because OK bisects BP, it also bisects FG and CD.

Prop. LXII. Theorem. If a straight line FEG, touch the hyperbola in E, the segments of it between the point of contact and the asymptotes will be equal, and if a straight line MN cut the hyperbola, or opposite hyperbolas, in K and L, the segments of it MK, LN, between the hyperbola and the asymptotes will be equal.



Draw the diameter OE, and its conjugate OD, and join DE, meeting the asymptote ON in R. Then EGOD is a parallelogram, and ER = RD, therefore EF = DO = EG.

Bisect KL in P, and draw the diameter OP, and through its vertex E, draw FG parallel to KL, it touches the hyperbola in E; therefore FE = EG, and, consequently, MP = PN. But the ordinate KL is bisected in P, or KP = PL; therefore MK = LN.

Cor. 1. The tangent FG = the diameter D*d* parallel to it.

Cor. 2. The rectangles $MK \times KN$, $ML \times LN$, $MK \times ML$ and $KN \times NL$, are all equal.

Cor. 3. The subtangent $FR = OR$, the distance from the centre.

Cor. 4. FD touches the adjacent hyperbola in D.

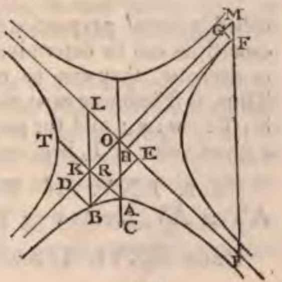
PROP. LXIII. Theorem. If a straight line which cuts the hyperbola, or the opposite hyperbolas, meets the asymptotes, the rectangle contained by the segments of it between a point in the hyperbola and the asymptotes, is equal to the square of the semidiameter parallel to it.

Let MN (see last figure) cut the hyperbola in K, and meet the asymptotes in M and N, and let DO be the semidiameter parallel to it. Draw OE the diameter conjugate to DO, it bisects KL; draw also FEG parallel to MN, it touches the hyperbola, and $EG = OD$. But $OE^2 : OD^2 = EG^2 :: OP^2 : PM^2$, and also $OE^2 : OD^2 :: OP^2 - OE^2 : PK^2$, therefore $OE^2 : OD^2 :: OE^2 : PM^2 - PK^2 = MK \times KN$, therefore $OD^2 = MK \times KN$.

Again, let KW cut the opposite hyperbolas, and meet the asymptotes in T and Y, and be parallel to the diameter OE, and let OD be its diameter which meets it in S, and let FH be the tangent parallel to it. Then $OD^2 : DF^2 = OE^2 :: OS^2 : ST^2$, and also $OD^2 : OE^2 :: OD^2 + OS^2 : KS^2$, therefore $OD^2 : OE^2 :: OD^2 : KS^2 - ST^2 = TK \times KY$, therefore $OD^2 = TK \times KY$.

Cor. The rectangles under segments of parallels between points in the hyperbola and the asymptotes are equal.

PROP. LXIV. Theorem. The rectangle contained by any two straight lines BD, BE, drawn from a point B in the hyperbola to the asymptotes, is equal to the rectangle contained by other two lines FG, FH, parallel to them, drawn to the same asymptotes from any point F of the four conjugate hyperbolas.



Through B and F draw any two parallels BKL and MFP. Then the triangles DBK, FGM, are similar, and also the triangles BEL and FHP, and therefore $BK : BD :: MF : FG$, and $BL : BE :: FP : FH$. Wherefore $BK \times BL : BD \times BE :: MF \times FP : GF \times FH$, and $BK \times BL = MF \times FP$; therefore $BD \times BE = GF \times FH$.

Cor. 1. If BD, FG, be parallel to the asymptote, the rectangle $DB \times DO = OG \times GF$, and if BE, FH, be also parallel to the asymptote, the parallelogram $DE = HG$.

Cor. 2. If AR be the line which joins the vertices of the axes, and C the focus, $AR = RO = \frac{1}{2} OC$; therefore the rectangle $OD \times BD = AR^2 = \frac{1}{4} OC^2$.

Cor. 3. If the hyperbolas be equilateral, or have their axes equal, the rectangle $OD \times DB = \frac{1}{2} OA^2$ (OA being the semiaxis.)

SECTION III.

OF VARIABLE QUANTITIES.

QUANTITIES which alter their values are called variable quantities. These are often so related to one another, that when one of them is increased, the others are increased or diminished according to a constant rule. Thus, if a body moves uniformly, the space it describes increases in the same ratio with the time; that is, if T and t be two times, and S and s the spaces run over in these times, then $T : t :: S : s$. This proportion is expressed generally thus, $T \propto S$, and read, the time is as the space.

If the quantities S, T, V be so related to one another, that when S is increased, both T and V are increased, so that their product has a constant ratio to S, then $S \propto TV$, read, S is as T and V jointly.

If these quantities be so related, that when V is increased, S is increased, and T diminished, so that their quotient has a constant ratio to V, then $V \propto \frac{S}{T}$. V is as S directly, and as T inversely. In this case, if S be constant, $V \propto \frac{1}{T}$. These are called *general proportions*, and if the values of the variable quantities can be determined at a given period of their increase or decrease, they can be reduced to determined proportions. Thus, if S becomes m at the same time that T becomes n, then $S : T :: m : n$; and the particular value of S, corresponding to a given value of T, is given.

PROP. LXV. If $T \propto V$, then $ST \propto SV$, and $AT \propto AV$, for $t : v :: T : V :: ST : SV :: AT : AV$.

PROP. LXVI. If $S \propto T$, and $V \propto X$, then $SV \propto TX$.

for $s:t::S:T$, and $v:x::V:X$; therefore $sv:tx::$ $SV:TX$.

Cor. Hence, if $S \propto T$, and $S \propto V$, then $S^2 \propto TV$, or $S \propto \sqrt{TV}$.

Prop. LXVII. If $S \propto T$, and $S \propto V$, then $S \propto T \pm V$, for $t:T::s:S::v:V$; therefore $s:S::t \pm v:T \pm V$ and $S \propto T \pm V$.

Cor. If $T \propto V$, then $T+V \propto T-V$, for $t:v::T:V$ and $t+v:t-v::T+V:T-V$; therefore $T+V \propto T-V$.

Prop. LXVIII. If $(V+T)^2 \propto (V-T)^2$, then $V^2+T^2 \propto VT$. For $(V+T)^2+(V-T)^2 \propto (V+T)^2-(V-T)^2$; that is, $V^2+T^2 \propto VT$.

Section IV.

LIMITS OF QUANTITIES AND RATIOS.

A constant quantity or ratio is said to be the *limit* of a variable one, when this latter can be altered, so as continually to approach to the constant one, and at length to come nearer to it than any other given quantity or ratio, but never to be equal to it.

A regular polygon is always less than the circle containing it, but increases as the number of its sides increase, and at length comes nearer to an equality with the circle than by any given difference. The circle is therefore said to be the limit of the polygon.

Also the perpendicular from the centre of the circle upon the side of the inscribed polygon is always less than the radius, but continually increases with the number of sides, and at length comes to be more nearly equal to the radius than by any given difference. The radius is therefore the limit of the perpendicular.

The tangent of an arc of a circle is always greater than its sine; but the ratio of the one to the other continually diminishes as the arc becomes less, and at length comes nearer to a ratio of equality than by any given difference. This ratio of equality is therefore the limit of that of the tangent to the sine.

Prop. LXIX. Let a and b be constant quantities, always greater than the variable quantities x and y, but

let x and y be capable of increase, so that $a-x$ shall be less than any given quantity; and also, that $b-y$ shall become less than any given quantity; the ratio of a to b is the limit of the ratio of x to y.

Let a be always to z as x to y. If the ratio of x to y be constant, then z is constant. If $z < b$, then y may be taken $> z$. But $x:y::a:z$, and $x < a$, therefore $y < z$; and it is also $> z$, which is impossible, therefore $a:b::x:y$.

If the ratio of x to y be variable, then z is variable; but its limit is constant, and cannot be less than b, as was proved before, neither can it be greater; for then if $y:x::b:v$, v would be less than a, which may be shown to be impossible as before; therefore the ratio of a to b is the limit of that of x to y.

In like manner, if x and y be always greater than a and b, but decrease so that $x-a$ and $y-b$ become less than any given quantities, it may be shown that the ratio of a to b is the limit of the ratio of x to y.

Let $x+y=2a$ to find the limit of xy, suppose x the greater $=a+v$, then $y=a-v$, and $xy=a^2-v^2$, as x or y approaches to $=a$, v becomes continually less, and is ultimately $=0$, therefore the limit of $xy=a^2$.

In like manner, if $x-y=2a$ by making $x=t+a$ and $y=t-a$, the limit of xy is $=a^2$.

Let $x+y=2a$ to find the limit of x^2+y^2. By proceeding as before, $x^2+y^2=(a+v)^2+(a-v)^2=2a^2-2v^2$, and as v^2 continually diminishes the ultimate value of $x^2+y^2=2a^2$; and the same will be the case if $x-y=2a$.

Let t be the increment of x to find the ratio of the limit of ax to that of x. When x increases to $x+t$, then ax increases to $ax+at$, and subtracting the first quantity ax, the increment is at; the ratio then is that of at to t, or of a to 1, which is independent of the value of x.

To find the ratio of the limit of the increment of ax^2 to that of x; when x becomes $x+t$, then ax^2 becomes $a(x+t)^2=ax^2+2axt+at^2$, and subtracting ax^2, the increment is $2axt+at^2$, which is to t as $2ax+at$ to 1; and when t becomes $=0$, the limiting ratio is $2ax$ to 1.

Prop. LXX. Let t be any increment of x, and v the corresponding increment of y. It is required to determine the limit of the ratio of the increments of the rectangles xy and ax.

When x becomes $x+t$, then y becomes $y+v$, and the rectangles become $(x+t)\times(y+v)$, and $a\times(x+t)$, and therefore

their increments are $(x+t)\times(y+v)-xy=xv+yt+tv$, and at. Let x be always to s as t to v, so that $xv=st$, then $xv+yt+tv:at::st+yt+vt:at$, or $::s+y+v:a$; and as v is continually diminishing, and at length becomes less than any given quantity, therefore the limit of the ratio of the increments is $s+y:a$, or if $\dot{x}:\dot{y}$ be the limit of $t:v$, it will be $s\dot{x}+y\dot{x}:a\dot{x}$, or since $s\dot{x}=x\dot{y}$, it will be $x\dot{y}+y\dot{x}:a\dot{x}$.

Cor. 1. If x decreases while y increases, the limiting ratio will be $x\dot{y}-y\dot{x}:a\dot{x}$.

If we divide the limit by the quantity, we get $\frac{x\dot{y}}{xy}+\frac{y\dot{x}}{yx}=\frac{\dot{y}}{y}+\frac{\dot{x}}{x}$ for the limiting ratio of the quantity. If x decreases, it is $\frac{\dot{y}}{y}-\frac{\dot{x}}{x}$.

Here $\dot{x}$ and $\dot{y}$ are the ultimate values of the increments of x and y.

Cor. 2. The limiting ratio of the increment of $xyz:a^2x$ is $xy\dot{z}+xz\dot{y}+yz\dot{x}:a^2\dot{x}$.

For let $yz=v$, then $xyz=xv$, and the limit $x\dot{v}+v\dot{x}:a^2\dot{x}$, but $\dot{v}=y\dot{z}+z\dot{y}$, and substituting $xy\dot{z}+xz\dot{y}+zy\dot{x}:a^2\dot{x}$.

The ratio of the limit of the increment of xyz to the quantity is $\frac{\dot{x}}{y}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}$. In the same manner, the ratio of $xyzv=\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}+\frac{\dot{v}}{v}$, &c.

Prop. LXXI. To find the limit of the ratio of the increment of x^2 to ax.

Suppose $x=y$, then $x^2=xy$, and the limiting ratio is $x\dot{y}+y\dot{x}:a\dot{x}$, or $2x\dot{x}:a\dot{x}$.

In like manner, it may be shown that the limit of the ratio of the increment of x^3 to a^2x is $3x^2\dot{x}:a^2\dot{x}$, and that of $x^4:a^3x$ is $4x^3\dot{x}$ to $a^3\dot{x}$, and so on; therefore the limit of the increment of x^n to that of $a^{n-1}x$ is $nx^{n-1}\dot{x}$ to $a^{n-1}\dot{x}$.

If the quantities x, y, z, v, (70. Cor. 2.) be equal, the ratio

will be $\frac{\dot{x}}{x}+\frac{\dot{x}}{x}+\frac{\dot{x}}{x}+\frac{\dot{x}}{x}$, &c. Suppose their number n, the sum is $\frac{n\dot{x}}{x}$, and this multiplied by x^n gives the limit $nx^{n-1}\dot{x}$.

PROP. LXXII. Required the limit of the ratio of the increment of $\frac{x}{y}$ to that $\frac{x}{a}$.

Let $\frac{x}{y}=z$, then $x=zy$, and the limit is $\dot{x}=z\dot{y}+y\dot{z}$; therefore $\dot{z}=\frac{\dot{x}-z\dot{y}}{y}$, and substituting for z, we have $\dot{z}=\frac{y\dot{x}-x\dot{y}}{y^2}$.

PROP. LXXIII. To find the limiting ratio of $(a+bx^n)^m$.

Let $a+bx^n=z$, then $z^m=(a+bx^n)^m$, and the ratio is $\frac{m\dot{z}}{z}$, but $\dot{z}=nbx^{n-1}\dot{x}$; therefore $\frac{m\dot{z}}{z}=\frac{mnbx^{n-1}\dot{x}}{a+bx^n}$, and the limit is $mnbx^{n-1}\dot{x}\times(a+bx^n)^{m-1}$.

NOTE. When constant quantities are connected with the variable ones by addition or subtraction, they disappear in the limits, and therefore the same quantity may be the limit of various quantities; but constant quantities with variable ones, by multiplication or division, retain their places.

It is of importance to distinguish between a quantity and its limit, and between the ratios of quantities and those of their limits, because they are not understood to be absolutely equal. We have called $\dot{x}$ the limit of x, and $\frac{\dot{x}}{x}$ the limiting ratio. The limit is called the *fluxion* in Britain, and the *differential* on the Continent, where it is marked dx.

RULES FOR FINDING THE FLUXIONS OF VARIABLE QUANTITIES.

RULE 1. To find the fluxion of a product.

Multiply the fluxion of each of the variable quantities into the product of all the rest, and the sum of the products thus obtained will be the fluxion required.

Thus the fluxion of xy is $x\dot{y}+y\dot{x}$, and the fluxion of $(a+x)\times(b+y)$ is $=(a+x)\dot{y}+(b+y)\dot{x}$.

RULE 2. To find the fluxion of any power.

Multiply the quantity by the index, and by the fluxion of the root, and diminish the index by unity.

Thus the fluxion of x^4 is $= 4x^3\dot{x}$, and the fluxion of ax^n is $= anx^{n-1}\dot{x}$.

RULE 3. To find the fluxion of a fraction.

From the fluxion of the numerator, multiplied by the denominator, take the fluxion of the denominator, multiplied by the numerator, and divide the remainder by the square of the denominator.

Thus the fluxion of $\frac{x}{y}$ is $=\frac{y\dot{x}-x\dot{y}}{y^2}$, and the fluxion of $\frac{a+x}{b+y}$ $=\frac{(b+y)\dot{x}-(a+x)\dot{y}}{(b+y)^2}$.

RULE 4. To find the limit or fluxion of a radical quantity, or of a compound power of it.

Take the limit of the quantity under the power or radical, and multiply it by the exponent of the power or radical. This divided by the quantity under the power or radical, gives the limiting ratio; and this ratio, multiplied by the given quantity, will be the fluxion.

EXAMPLES.

1. Required the limit of $(a+bx+cx^2)^{\frac{1}{2}}$.

Let $z=a+bx+cx^2$, then $z^{\frac{1}{2}}=(a+bx+cx^2)^{\frac{1}{2}}$, and the ratio of $z^{\frac{1}{2}}$ is $\frac{\frac{1}{2}\dot{z}}{z}$ or $\frac{\dot{z}}{2z}$ but $\dot{z}=b\dot{x}+2cx\dot{x}$; therefore $\frac{\dot{z}}{2z}=$ $=\frac{b\dot{x}+2cx\dot{x}}{2(a+bx+cx^2)}$, and the limit is $\frac{(b\dot{x}+2cx\dot{x})\times(a+bx+2cx^2)^{-\frac{1}{2}}}{2}$, or $\frac{b\dot{x}+2cx\dot{x}}{2(a+bx+cx^2)^{\frac{1}{2}}}$.

2. Required the limit of $(a+bx+cx^2+dx^3)^4$.

Let $z=a+bx+cx^2+dx^3$, then $z^4=(a+bx+cx^2+dx^3)^4$, and $\frac{4\dot{z}}{z}$ is its ratio, but $\dot{z}=b\dot{x}+2cx\dot{x}+3dx^2\dot{x}$; therefore $\frac{4\dot{z}}{z}=\frac{4b\dot{x}+8cx\dot{x}+12dx^2\dot{x}}{a+bx+cx^2+dx^3}$, and the limit is $(4b\dot{x}+8cx\dot{x}+12dx^2\dot{x})\times(a+bx+cx^2+dx^3)^3$.

3. Let $u = x(a^2+x^2)(a^2-x^2)^{\frac{1}{2}}$; hence the limiting r

is $\frac{\dot{u}}{u} = \frac{\dot{x}}{x} + \frac{2x\dot{x}}{a^2+x^2} - \frac{x\dot{x}}{a^2-x^2}$, which, reduced to a common

nominator and added, is $\frac{(a^4+a^2x^2-4x^4)\dot{x}}{x(a^2+x^2)(a^2-x^2)}$; this, multipl

by the given quantity, gives the fluxion $\dot{u} = \frac{(a^4+a^2x^2-4x^4)\dot{x}}{(a^2-x^2)^{\frac{1}{2}}}$

4. Let $u = \frac{a^2-x^2}{a^4+a^2x^2+x^4}$; the limit is

$\frac{-2x\dot{x}(a^4+a^2x^2+x^4)-2x\dot{x}(a^2+2x^2)(a^2-x^2)}{(a^4+a^2x^2+x^4)^2}$, which, reduced

and added, becomes $\dot{u} = \frac{-2x\dot{x}(2a^4+2a^2x^2-x^4)}{(a^4+a^2x^2+x^4)^2}$.

5. Let $u = (a-bx^{-\frac{1}{2}}+(c^2-x^2)^{\frac{2}{3}})^{\frac{3}{4}}$, putting $y = bx^{-\frac{1}{2}}$

and $z = (c^2-x^2)^{\frac{2}{3}}$, then $\dot{y} = -\frac{1}{2}bx^{-\frac{3}{2}}\dot{x}$ and $\dot{z} = -\frac{4}{3}x\dot{x}$

$\times(c^2-x^2)^{-\frac{1}{3}}$; hence $\dot{u} = \frac{+3b(c^2-x^2)^{\frac{1}{3}}-8x^{\frac{5}{2}})\dot{x}}{8x^{\frac{3}{2}}(c^2-x^2)^{\frac{1}{3}}(a-bx^{-\frac{1}{2}}+(c^2-x^2)^{\frac{2}{3}})^{\frac{1}{4}}}$

QUANTITIES.	RATIOS.	FLUXIONS.
1. x^2y^3	$\frac{2\dot{x}}{x}+\frac{3\dot{y}}{y}$	$2xy^3\dot{x}+3x^2y^2\dot{y}$
2. $3v^5$	$\frac{5\dot{v}}{v}$	$15v^4\dot{v}$
3. $\frac{2x^{\frac{9}{7}}}{5}$	$\frac{9\dot{x}}{7x}$	$\frac{18x^{\frac{2}{7}}\dot{x}}{35}$
4. $(a^2+x^2)^3$	$\frac{6x\dot{x}}{(a^2+x^2)}$	$(a^2+x^2)^2\times 6x\dot{x}$
5. $(a^2+x^2)^{\frac{1}{2}}$	$\frac{x\dot{x}}{a^2+x^2}$	$\frac{x\dot{x}}{\sqrt{a^2+x^2}}$
6. $(a^5+x^5)^{\frac{3}{2}}$	$\frac{15x^4\dot{x}}{2(a^5+x^5)}$	$\frac{15x^4\dot{x}}{2}(a^5+x^5)^{\frac{1}{2}}$
7. $(x+y)^2$	$\frac{2\dot{x}+2\dot{y}}{x+y}$	$2(x+y)\times(\dot{x}+\dot{y})$
8. $(x^2+y^2)^{\frac{3}{2}}$	$\frac{3x\dot{x}+3y\dot{y}}{x^2+y^2}$	$(x\dot{x}+y\dot{y})\times 3\sqrt{x^2+y^2}$

	QUANTITIES.	RATIOS.	FLUXIONS.
9.	$\frac{1}{\sqrt[9]{(a^2+x^2)^5}}$	$\frac{-10x\dot{x}}{9(a^2+x^2)}$	$\frac{-10x\dot{x}}{9(a^2+x^2)^{1\frac{4}{5}}}$
10.	$\frac{a}{x^n}$	$\frac{-n\dot{x}}{x}$	$\frac{-na\dot{x}}{x^{n+1}}$
11.	$x^{\frac{5}{3}}y^{\frac{7}{2}}z$	$\frac{5\dot{x}}{3x}+\frac{7\dot{y}}{2y}+\frac{\dot{z}}{z}$	$\frac{5}{3}x^{\frac{2}{3}}y^{\frac{7}{2}}z\dot{x}+\frac{7}{2}x^{\frac{5}{3}}y^{\frac{5}{2}}z\dot{y}+x^{\frac{5}{3}}y^{\frac{7}{2}}\dot{z}$
12.	$\frac{x^2}{y^3}$	$\frac{2\dot{x}}{x}-\frac{3\dot{y}}{y}$	$\frac{2xy\dot{x}-3x^2\dot{y}}{y^4}$
13.	$\frac{x+y}{z^3}$	$\frac{\dot{x}+\dot{y}}{x+y}-\frac{3\dot{z}}{z}$	$\frac{z(\dot{x}+\dot{y})-(x+y)3\dot{z}}{z^4}$
14.	$\frac{xy}{z^2}$	$\frac{\dot{x}}{x}+\frac{\dot{y}}{y}-\frac{2\dot{z}}{z}$	$\frac{z(x\dot{y}+y\dot{x})-2xy\dot{z}}{z^3}$
15.	$x^2(a^4+y^4)^{\frac{3}{2}}$		$6x^2y^3\dot{y}(a^4+y^4)^{\frac{1}{2}}+2x\dot{x}(a^4+y^4)^{\frac{3}{2}}$
16.	$(a^2+x^2)^{\frac{1}{2}}\times(b^2+v^2)^{\frac{1}{2}}$		$\frac{v\dot{v}(a^2+x^2)^{\frac{1}{2}}}{\sqrt{b^2+v^2}}+\frac{x\dot{x}(b^2+v^2)^{\frac{1}{2}}}{\sqrt{a^2+x^2}}$
17.	$(x^2+y^3+z^4)^{\frac{7}{3}}$		$\frac{7}{3}(2x\dot{x}+3y^2\dot{y}+4z^3\dot{z})\times(x^2+y^3+z^4)^{\frac{4}{3}}$
18.	$\sqrt{x^2+\sqrt{a^2+y^2}}$		$\frac{2x\dot{x}-y\dot{y}(a^2+y^2)^{-\frac{1}{2}}}{2\sqrt{x^2+\sqrt{a^2+y^2}}}$

The limit of the quantity is here given, and the ratio of that limit to the quantity; but the ratio of the limit of the quantity to that of the variable is frequently used, and is called the coefficient limit, being independent of the limit $\dot{x}$. Thus, if $u=x^n$, then is $\dot{u}=nx^{n-1}\dot{x}$, and $\frac{\dot{u}}{\dot{x}}=nx^{n-1}$ is called the coefficient limit.

OF SECOND, THIRD, &c. LIMITS.

Though the limit or fluxion of a quantity simply considered is constant, yet as the variable quantities may alter their state and their ratios, this alteration may affect the fluxion, which in this case may be a variable quantity, and therefore have itself a fluxion; the fluxion is found from it by the preceding rules. In the same manner its fluxion may be variable, and thus have a fluxion. These fluxions are commonly referred to the original quantity, and are called its second, third, &c. fluxions, and are marked with dots above them according to

their order; thus $\ddot{x}$ is the second fluxion of x, $\dddot{x}$ its third fluxion, &c.

If the $\dot{x}$ be constant, x^n will have n fluxions and no more, n being an affirmative whole number. For the first fluxion is $nx^{n-1}\dot{x}$; and x only being variable, its fluxion is $n.(n-1)\times x^{n-2}\dot{x}^2$; and the fluxion of this is $n.(n-1).(n-2).x^{n-3}\dot{x}^3$, &c. So that when we have taken the fluxion n times, the index of x becomes $=0$, and $x^0=1$; hence the fluxion then becomes $n.(n-1)$ . . . $2.1.\dot{x}^n$, which, being a constant quantity, can have no further fluxion.

OF LOGARITHMS.

Prop. LXXIV. Let x, x^2, &c. x^n, x^{n+1} be a geometrical series, the logarithms of its terms form an arithmetical series; that is, their increments are all equal, or they have a constant limit.

The increment of x^n is $x^{n+1}-x^n=x^n\times\overline{x-1}$, and its ratio to the term 1 is $x^n(x-1)\div x^n=x-1$, which is constant in the same series, also $\overline{\dot{x^n}}|\div x^n=\frac{n\dot{x}}{x}$; therefore $\overline{\dot{x-1}}|=\frac{\dot{x}}{x}$. Let $\dot{X}$ be the constant fluxion of the logarithms, this will be the fluxion of the geometrical series at the beginning, or $\dot{X}=\frac{A\dot{x}}{x}$; that is, the fluxion of the logarithm has a constant ratio to the fluxion of the number divided by that number. Thus the fluxion of the logarithm of $x\pm a$ is $\frac{\dot{x}}{x\pm a}$.

1. Let $y^x=z$, and let Y, Z be the logarithms of y, z, then $x\text{Y}=\text{Z}$, and their fluxions will be $x\dot{\text{Y}}+\text{Y}\dot{x}=\dot{\text{Z}}$, but $\dot{\text{Y}}=\frac{\dot{y}}{y}$, and $\dot{\text{Z}}=\frac{\dot{z}}{z}$; therefore $\frac{x\dot{y}}{y}+\text{Y}\dot{x}=\frac{\dot{z}}{z}$, and $\dot{z}=\frac{xz\dot{y}}{y}+\text{Y}z\dot{x}=xy^{x-1}\dot{y}+y^x\text{Y}\dot{x}$.

2. Let $y=\text{X}^n$ (X the log. of x) then $\dot{y}=n\text{X}^{n-1}\dot{\text{X}}=n\text{X}^{n-1}\frac{\dot{x}}{x}$.

3. Let $y=x^n\text{X}^m$, then $\dot{y}=\text{X}^{m-1}x^{n-1}\dot{x}(n\text{X}+m)$.

4. Required the fluxion of the logarithm of $\frac{a+x}{a-x}$. The fluxion of the number is $\frac{\dot{x}(a-x)+\dot{x}(a+x)}{(a-x)^2}=\frac{2a\dot{x}}{(a-x)^2}$, and, dividing this by $\frac{a+x}{a-x}$, we obtain $\frac{2a\dot{x}}{a^2-x^2}$ for the fluxion of the logarithm.

The most useful forms of the fluxions of logarithms are the following:—

1. $\frac{a+x}{a-x}$. The fluxion of its logarithm is $\frac{2a\dot{x}}{a^2-x^2}$

2. $\frac{x-a}{x+a}$ . . . . $\frac{2a\dot{x}}{x^2-a^2}$

3. $x+(x^2+a^2)^{\frac{1}{2}}$ . . . $\frac{\dot{x}}{(x^2+a^2)^{\frac{1}{2}}}$

4. $a^2+x^2+x(2a^2+x^2)^{\frac{1}{2}}$ . . $\frac{2\dot{x}}{(2a^2+x^2)^{\frac{1}{2}}}$

5. $\frac{a-(a^2\pm x^2)^{\frac{1}{2}}}{a+(a^2\pm x^2)^{\frac{1}{2}}}$ . . . $\frac{2a\dot{x}}{x(a^2\pm x^2)^{\frac{1}{2}}}$

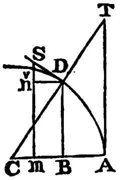
6. $\frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}}$ . . . $\frac{\dot{x}}{x(1-x^2)^{\frac{1}{2}}}$

7. $\frac{(x\sqrt{-1}+1-x^2)^{\frac{1}{2}}}{\sqrt{-1}}$ . . . $\frac{\dot{x}}{(x^2-1)^{\frac{1}{2}}}$

8. $\frac{(x^2+1)^{\frac{1}{2}}+x}{(x^2+1)^{\frac{1}{2}}-x}$ . . . . $\frac{\dot{x}}{(x^2+1)^{\frac{1}{2}}}$

OF CIRCULAR ARCS.

PROB. LXXV.—To express the fluxions of circular arcs in terms of the sine, tangent, secant, &c.



Let the radius AC be $=r$, the versed sine AB $=x$, the sine BD $=y$, the tangent AT $=t$, the secant CT $=s$, and the arc AD $=v$. Draw the tangent Ds, and the line sm parallel to BD, and Dn parallel to AC, and let sm meet the arc in v, then $ns > nv$. Therefore the ratio of Dn to nv is always greater than that of Dn to ns, but by diminishing Dn it continu-

ally approaches to that ratio, and at length comes nearer to it than any given ratio greater than that of Dn to ns; therefore the ratio of Dn to ns is the limit or fluxion of the ratio of Dn to nv, and of course Dn : Ds is the fluxion of the ratio of Dn : Dv. But the triangles nDs, CDB are similar, for CDs being a right angle, nDs = BDC; therefore BD : DC : : nD : Ds, and $n\mathrm{D} = \dot{x}$, and $\mathrm{D}s = \dot{v}$; therefore $y : r :: \dot{x} : \dot{v} = \frac{r\dot{x}}{y}$. In like manner, BC : CD : : ns : sD, and $ns = \dot{y}$; therefore $r - x : r :: \dot{y} : \dot{v} = \frac{r\dot{y}}{r-x}$, now $r - x = \sqrt{r^2 - y^2}$, and $y = \sqrt{2rx - x^2}$, therefore $\dot{v} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r\dot{x}}{\sqrt{2rx - x^2}}$. Again, CB : BD : : CA : AT, or $r - x : y : r : t = \frac{ry}{r-x}$, whence $\dot{t} = \frac{r^3\dot{x}}{y \times (r-x)^2} = \frac{r^2\dot{v}}{(r-x)^2}$ $= \frac{r^2 + t^2}{r^2}\dot{v}$; also CB : CD : : CA : CT, or $r - x : r :: r : s = \frac{r^2}{r-x}$, and $\dot{s} = \frac{r^2\dot{x}}{(r-x)^2}$; therefore $\dot{s} : \dot{t} :: y : r :: \dot{x} : \dot{v}$, whence again $\dot{v} = \frac{r^2\dot{s}}{st} = \frac{r^2\dot{s}}{s\sqrt{s^2 - r^2}} = \frac{r^2\dot{t}}{s^2} = \frac{r^2\dot{t}}{r^2 + t^2}$. If $r = 1$, then $\dot{v} = \frac{\dot{y}}{\sqrt{1 - y^2}} = \frac{\dot{x}}{x(1-x)^{\frac{1}{2}}} = \frac{\dot{t}}{1 + t^2} = \frac{\dot{s}}{s(s^2 - 1)^{\frac{1}{2}}}$.

These are the most useful forms of fluxions of circular arcs.

TO FIND THE SINE AND COSINE OF AN ARC v.

Assume sin. $v = av + bv^2 + cv^3$, &c. and cos. $v = 1 + mv + nv^2 + pv^3$, &c. then $\overline{\text{sin. } v}|^{\cdot} = a\dot{v} + 2bv\dot{v} + 3cv^2\dot{v}$, &c. and $\overline{\text{cos. } v}|^{\cdot} = m\dot{v} + 2nv\dot{v} + 3pv^2\dot{v}$, &c. but $\overline{\text{sin. } v}|^{\cdot} = \dot{v}$ cos. v, and $\overline{\text{cos. } v}|^{\cdot} = -\dot{v}$ sin. v, whence we have two equations $a + 2bv + 3cv^2$, &c. $= 1 + mv + nv^2 + pv^3$, &c. and $av + bv^2 + cv^3$, &c. $= -m - 2nv - 3pv^2$, &c. and equating the coefficients, we have $a = 1$, $-m = o$, $b = o$, $n = -\frac{1}{2}$, $c = \frac{-1}{2 \cdot 3}$, $p = o$, $d = o$, $q = \frac{+1}{2 \cdot 3 \cdot 4}$, $e = \frac{+1}{2 \cdot 3 \cdot 4 \cdot 5}$, &c.; therefore, substituting these values,

we get sin. $v = v - \frac{v^3}{2\cdot3} + \frac{v^5}{2\cdot3\cdot4\cdot5} - \frac{v^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}$, &c. Cos. $v = 1 - \frac{v^2}{2} + \frac{v^4}{2\cdot3\cdot4} - \frac{v^6}{2\cdot3\cdot4\cdot5\cdot6} + \frac{v^8}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}$, &c.

TO FIND THE LENGTH OF THE ARC OF WHICH THE TANGENT IS t.

Assume $v = at + bt^2 + ct^3$, &c. then $\dot{v} = a\dot{t} + 2bt\dot{t} + 3ct^2\dot{t}$, &c. But $\dot{v} = \frac{\dot{t}}{1+t^2} = \dot{t} - t^2\dot{t} + t^4\,\dot{t} - t^6\dot{t}$, and, equating the coefficients, we have $a = 1$, $b = o$, $c = \frac{-1}{3}$, $d = o$, $e = \frac{+1}{5}$, &c.; therefore $v = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}$, &c.

OF FLUENTS OR INTEGRALS.

PROP. LXXVI. The limit or fluxion of any quantity may be found by the preceding rules; but it is often difficult to find the quantity which will produce a given fluxional expression. This quantity is called the fluent or integral: The following are the most general and simple rules for finding fluents.

RULE 1. If the quantity be simple, and have one variable, add unity to the index, and divide by the increased index, and by the fluxion of the root.

Thus, because the limit of x^{n+1} is $(n+1)x^n\dot{x}$, therefore the quantity of which $x^n\dot{x}$ is the limit will be $\frac{x^{n+1}\dot{x}}{(n+1)\dot{x}}$.

RULE 2. If the quantity be a compound power or radical, and the quantity without the vinculum have a given ratio to the fluxion of the quantity under the vinculum, the fluent may be found by the preceding rule.

Because the limit of $(2ax - x^2)^{\frac{1}{2}}$ is $\frac{1}{2}\dot{x}\,(2a - 2x)(2ax - x^2)^{\frac{1}{2}-1}$, therefore the integral of $\dot{x}\ (a - x)\ (2ax - x^2)^{-\frac{1}{2}}$ is

$$\frac{(2ax - x^2)^{-\frac{1}{2}+1}(a - x)\dot{x}}{\frac{1}{2}(a - x)\,2\dot{x}} = (2ax - x^2)^{\frac{1}{2}}.$$

RULE 3. If the quantity consists of as many terms as there

are variable quantities, and each term be the product of the fluxion of one of the variables by all the rest. Take the fluent of any term, upon the supposition of all the quantities being constant, except that which has its fluxion in it, and it will be the fluent of the whole.

Because the fluxion of xvz is $vz\dot{x}+xz\dot{v}+xv\dot{z}$, the whole integral of $vz\dot{x}+xz\dot{v}+xv\dot{z}$ will be xvz.

And because the limit of the logarithm of $\frac{x-1}{x+1}$ is $\frac{2\dot{x}}{x^2-1}$, therefore if a limit occur of the form $\frac{\dot{x}}{x^2-1}$, we know that its integral is $\frac{1}{2}$ log. $\frac{x-1}{x+1}$. Also the limit of the arc of which the tangent is x, is $\frac{\dot{x}}{x^2+1}$; therefore the number of which $\frac{\dot{x}}{x^2+1}$ is the fluxion, is the arc of which the tangent is x: Thus we may determine fluents when the limit is of any form mentioned in logarithms or arcs of the circle.

RULE 4. If the quantity be a compound power, or radical, and the index of the variable without the vinculum increased by one be a multiple of that within it, the power, or radical, may be expanded into a series, and multiplied by the quantity without the vinculum, and then the fluent of each term may be found separately. Or a letter may be taken for the quantity under the vinculum, and the whole expressed in terms of that letter and expanded, which will be often more simple than the other, and the fluent of each term is to be taken as before. When the exponent of x without the power, or radical, increased by 1, is a multiplier of that within it, the expansion will consist of a finite number of terms, and some of these may be limits of logarithmic or circular functions.

If $\dot{y}=\frac{\dot{x}}{(1-x^2)^{\frac{1}{2}}}$, by expanding $(1-x^2)^{-\frac{1}{2}}$ it becomes $1+\frac{x^2}{2}+\frac{3x^4}{2\cdot4}+\frac{3\cdot5x^6}{2\cdot4\cdot6}$, &c.; therefore $\dot{y}=\dot{x}+\frac{x^2\dot{x}}{2}+\frac{3x^4\dot{x}}{2\cdot4}+\frac{3\cdot5x^6\dot{x}}{2\cdot4.6}$, &c.; and, taking the fluent of each term, we have $y=x+\frac{x^3}{2\cdot3}+\frac{3x^5}{2\cdot4\cdot5}+\frac{3\cdot5x^7}{2\cdot4\cdot6\cdot7}$, &c.

NOTE. Constant quantities connected with the variable ones, by addition or subtraction, disappear in taking the

fluxion; it is necessary to restore these when the fluent is taken. Consider whether the fluent becomes $= o$, or to some known quantity at the time it ought; if not, annex to it such a constant quantity as will make it $=$ its proper value. We commonly annex C for this constant, the value of which may be determined afterwards.

FLUXIONS.	FLUENTS.
$x^n \dot{x}$	$\frac{x^{n+1.}}{n+1}$
$\frac{x\dot{x}}{\sqrt{a^2+x^2}}$	$\sqrt{a^2+x^2}$
$x\dot{y}+y\dot{x}$	xy
$(a^n+x^n)^m x^{n-1}\dot{x}$	$\frac{(a^n+x^n)^{m+1}}{n(m+1)}$
$(a^2+x^2)^5 x^3 \dot{x}$	$\frac{6x^2-a^2}{84}(a^2+x^2)^6$
$\frac{2a\dot{x}}{a^2-x^2}$	$\mathrm{L}\frac{a+x}{a-x}$
$\frac{\dot{x}}{(x^2\pm a^2)^{\frac{1}{2}}}$	$\mathrm{L}\,x+\sqrt{x^2\pm a^2}$
$\frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$	$\mathrm{L}\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$

L means the hyperbolic logarithm, or the common logarithm multiplied by 2·302585.

SECTION V.

OF THE LENGTHS AND AREAS OF CURVES.

PROP. LXXVII. Problem. To determine the length of any curve ABC.

Let AE $= x$, EB $= y$, and the curve AB $= z$. Draw GF parallel to BE and BG to touch the curve at B, and BL parallel to AD. Then, while AE has increased to AF, BE has increased to FH, and the tangent is BG, and GL is always greater than LH. But as BL decreases, GL becomes more nearly equal to HL, and at length

they will become more nearly equal than by any given difference; therefore, representing BL by $\dot{x}$, LG by $\dot{y}$, and BG by $\dot{z}$, we have $\dot{z}^2=\dot{x}^2+\dot{y}^2$, or $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$. And the fluent of this equation will be the value of z, which fluent must be determined from the nature of the curve.

EXAMPLES.

1. Let the curve be a parabola, of which the principal vertex is A, and $p=$ the parameter, then $px=y^2$, and $p\dot{x}=2y\dot{y}$, and $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{\frac{1}{2}p}\sqrt{\frac{1}{4}p^2+y^2}=\frac{2y^2\dot{y}+2d^2\dot{y}}{2d\sqrt{y^2+d^2}}(d=\frac{1}{2}p)=$ $\frac{2y^3\dot{y}+2d^2y\dot{y}}{2d\sqrt{y^4+y^2d^2}}=\frac{2y^3\dot{y}+d^2y\dot{y}}{2d\sqrt{y^4+y^2d^2}}+\frac{d^2y\dot{y}}{2d\sqrt{y^4+y^2d^2}}$. Here the fluent of the first term is $\frac{y\sqrt{y^2+d^2}}{2d}$, and that of the second is $\frac{1}{2}d\times$ hyp. log. of $\frac{y+\sqrt{y^2+d^2}}{d}$; therefore the length of the curve is $\frac{y\sqrt{y^2+d^2}}{2d}+\frac{1}{2}d\times\log.\frac{y+\sqrt{y^2+d^2}}{d}$.

2. Let the curve be a circle, then $\dot{z}=\frac{\dot{x}}{y}$ (see Prop. 75.), which, being reduced to a series, and the fluent taken, becomes $2y\times(\frac{1}{2}+\frac{x^2}{3y^2}-\frac{x^4}{2\cdot3y^4}+\frac{x^6}{5\cdot7y^6}-\frac{x^8}{7\cdot9y^8}$, &c.) or putting $v^2=\frac{x^2}{y^2}$, it becomes for the arc of which the chord is $2y$. $=4y\times(\frac{1}{2}+\frac{1}{3}v^2-\frac{v^4}{2\cdot3}+\frac{v^6}{5\cdot7}-\frac{v^8}{7\cdot9}$, &c.) and this series is nearly equal to $2y\,\frac{15+13v^2}{15+3v^2}$, but more nearly equal to $\frac{4y}{3}\times$ $\left(\frac{3}{2}+v^2-v^4\times\frac{\frac{1}{3}v^2+1}{\frac{5}{9}v^2+1}\right)$, which are the two approximations given in Prob. 17, Mensuration of Superficies.

Otherwise, by Prop. 75, $\dot{z}=\frac{\dot{t}}{1+t^2}$, which, reduced to a series, becomes $\dot{t}-t^2\dot{t}+t^4\dot{t}-t^6\dot{t}$, &c.; and the fluent being

taken, $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7$, &c. This is exemplified in Prob. 13, Mensuration of Superficies.

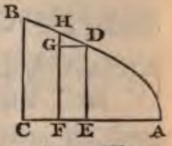
3. Let the curve be an ellipse, then by the 11th Formula, Prop. 54, $y = \frac{b}{a} \times \sqrt{2ax - x^2} = \frac{b}{a}\sqrt{a^2 - v^2}$ (v = distance of the ordinate from the centre $= a - x$); therefore $\dot{y} = \frac{-bv\dot{v}}{a\sqrt{a^2 - v^2}}$, and therefore $\dot{z} = \frac{\dot{v}\sqrt{\left(a^2 - \frac{a^2 - b^2}{a^2}v^2\right)}}{\sqrt{a^2 - v^2}}$, or (putting $d = 1 - \frac{b^2}{a^2}$), $\dot{z} = \frac{a\dot{v}\sqrt{a^2 - dv^2}}{\sqrt{a^2 - v^2}} = \frac{a\dot{v}}{\sqrt{a^2 - v^2}} \times \left(1 - \frac{dv^2}{2a^2} - \frac{d^2v^4}{2 \cdot 4a^4} - \frac{3d^3v^6}{2 \cdot 4 \cdot 6a^6}, \text{\&c.}\right)$ by throwing $\sqrt{a^2 - dv^2}$ into a series. But the fluent of $\frac{a\dot{v}}{\sqrt{a^2 - v^2}}$, is the corresponding arc of the circle, and therefore the whole fluent (putting $t^2 = a^2 - v^2$) is $A - \frac{d}{2a^2} \times \frac{a^2A - tv}{2} - \frac{d^2}{2 \cdot 4a^4} \times \frac{3a^2B - tv^3}{4} - \frac{3d^3}{2 \cdot 4 \cdot 6a^6} \times \frac{5a^2C - tv^5}{6}$, &c. where $B = \frac{a^2A - tv}{2}$, $C = \frac{3a^2B - tv^3}{4}$, $D = \frac{5a^2C - tv^5}{6}$.

If the whole quadrant be required, $v = a$, and $t = o$, and then $z = A \times \left(1 - \frac{d}{2 \cdot 2} - \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}, \text{\&c.}\right)$ and this is nearly the fourth part of the series to which the rule in Prob. 26, Mensuration of Superficies, might be reduced.

4. Let the curve be a hyperbola, then $y = \frac{b}{a}\sqrt{ax + x^2}$, whence $x = \frac{a}{b} \times \sqrt{b^2 + y^2} - a$, and $\dot{x} = \frac{ay\dot{y}}{b\sqrt{b^2 + y^2}}$, whence $\dot{z}$ $= \dot{y}\frac{\sqrt{b^2 + \frac{b^2 + a^2}{b^2}y^2}}{\sqrt{b^2 + y^2}}$, or (putting $q = \frac{b^2 + a^2}{b^4}$) $\dot{z} = \frac{b\dot{y}}{\sqrt{b^2 + y^2}} \times \sqrt{1 + qy^2} = \frac{b\dot{y}}{\sqrt{b^2 + y^2}} \times \left(1 + \frac{qy^2}{2} + \frac{q^2y^4}{2 \cdot 4} + \frac{3q^3y^6}{2 \cdot 4 \cdot 6} + \frac{3 \cdot 5q^4y^3}{2 \cdot 4 \cdot 6 \cdot 8}, \text{\&c.}\right)$ Now the fluent of $\frac{b\dot{y}}{\sqrt{b^2 + y^2}} = b \times \text{hyp.log.of}\,\frac{y + \sqrt{b^2 + y^2}}{b}$

$= \mathrm{A}$, and therefore $z = b \times \Big(\mathrm{A} + \frac{q\mathrm{B}}{2} - \frac{q^2\mathrm{C}}{2\cdot 4} + \frac{3q^3}{2\cdot 4\cdot 6}\mathrm{D} + \frac{3\cdot 5q^4}{2\cdot 4\cdot 6\cdot 8}$ E, &c.$\Big)$ where $\mathrm{B} = \frac{y\sqrt{b^2+y^2} - b^2\mathrm{A}}{2}$, $\mathrm{C} = \frac{y^3\sqrt{b^2+y^2} - 3b^2\mathrm{B}}{4}$, $\mathrm{D} = \frac{y^5\sqrt{b^2+y^2} - 5b^2\mathrm{C}}{6}$, &c.

Prop. LXXVIII. Problem. To find the area of a curvilinear figure.



Let $\mathrm{AE} = x$, and $\mathrm{ED} = y$. Draw HF parallel to DE, and DG to AC. The parallelogram GFED is always less than the curvilinear HFED, but it continually approaches to an equality with it, as HF approaches to DE, and at length would differ from it by a quantity less than any given quantity. Therefore GE is the limit of the increment of HFED; that is, $y\dot{x}$ is the fluxion of the area AED, and its fluent found from the nature of the curve, and properly corrected, will be the area.

EXAMPLES.

1. Let the curve be a parabola, and p the parameter, then $px = y^2$; therefore $p\dot{x} = 2y\dot{y}$, and $y\dot{x} = \frac{2y^2\dot{y}}{p}$, and the fluent of this or the area $= \frac{2y^3}{3p} = \frac{2xy}{3}$, which is the rule in Prob. 27, Mensuration of Superficies.

If X be another abscissa and Y its ordinate, and $\mathrm{X} - x = d$, then $\mathrm{Y}^2 : \mathrm{Y}^2 - y^2 :: \mathrm{X} : d :: \mathrm{X}p : dp$, and $\mathrm{X}p = \mathrm{Y}^2$; therefore $dp = \mathrm{Y}^2 - y^2$ and $p = \frac{\mathrm{Y}^2 - y^2}{d}$, and the area of the frustum is $\frac{2}{3}\left(\frac{\mathrm{Y}^3 - y^3}{p}\right) = \frac{2}{3}d\left(\frac{\mathrm{Y}^3 - y^3}{\mathrm{Y}^2 - y^2}\right) = \frac{2}{3}d\left(\frac{\mathrm{Y}^2 + \mathrm{Y}y + y^2}{\mathrm{Y} + y}\right)$ $= \frac{2}{3}d\left(\mathrm{Y} + \frac{y^2}{\mathrm{Y}+y}\right)$, which is the rule in Prob. 28, Mensuration of Superficies.

2. Let the curve be the segment of a circle, of which the radius is r, then $2rx - x^2 = y^2$; therefore $\dot{x} =$

$\frac{\dot{y}y}{r-x} = \frac{y\dot{y}}{\sqrt{r^2-y^2}}$, and $y\dot{x} = \frac{y^2\dot{y}}{\sqrt{r^2-y^2}}$, which, being reduced to a series, and the fluent taken, becomes $2xy \times \left(\frac{1}{3} + \frac{x^2}{3 \cdot 5y^2} - \frac{x^4}{3 \cdot 5 \cdot 7y^4} + \frac{x^6}{5 \cdot 7 \cdot 9y^6}, \text{ \&c.}\right)$ a series which coincides very nearly with $\frac{2yx}{15} \times \left(5 + q^2 - \frac{q^4}{21} \times \frac{4q^2+33}{5q^2+11}\right)$, supposing $q = \frac{x}{y}$. This is the second approximation in Prob. 22, Mensuration of Superficies.

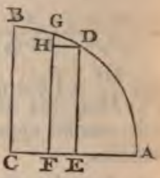
3. Let the curve be an ellipse, of which the semiaxes are a and b; then by the 11th Formula, Prop. 54, $y^2 = \frac{b}{a} \times (2ax - x^2)$, which is the equation for the circle multiplied by $\frac{b}{a}$; therefore the area of the circle, or of any portion of it, multiplied by $\frac{b}{a}$, will give the ellipse, or a similar portion of it, as in Prob. 25, Mensuration of Superficies.

4. Let the curve be a hyperbola, of which the semiaxes are a and b, then the equation is $\frac{b^2}{a^2} \times (2ax + x^2) = y^2$, or taking $v = a + x$, then $\frac{b}{a}\sqrt{v^2 - a^2} = y$, and $y\dot{v} = \frac{b\dot{v}}{a}\sqrt{v^2 - a^2}$, and the fluent of its double is $\frac{b}{a}\sqrt{v^2 - a^2} - ab \times \text{hyp. log. of } \frac{v + \sqrt{v^2 - a^2}}{a} = vy - ab \times \text{hyp. log. of } \frac{ay + bv}{ab}$, which is the rule in Prob. 29 of Mensuration of Superficies.

5. To find the area between the hyperbola and the asymptotes (see figure to Prop. 64.).

Let $OR = RA = c$, $RD = x$, and $DB = y$, then $OD = c + x$, and $OD \times DB = OR \times RA$; therefore $y = \frac{c^2}{c+x} = c - x + \frac{x^2}{c} - \frac{x^3}{c^2}$, &c. and $y\dot{x} = c\dot{x} - x\dot{x} + \frac{x^2\dot{x}}{c} - \frac{x^3\dot{x}}{c^2}$, &c.; therefore the area $RABD = c^2 \times \left(\frac{x}{c} - \frac{x^2}{2c^2} + \frac{x^3}{3c^3} - \frac{x^4}{4c^4}\right)$, &c. $= c^2 \times \text{hyp. log. } \frac{c+x}{c}$.

Prop. LXXIX. Problem. To find the surface of a solid generated by the revolution of a curve about an axis.



Let the curve ADB revolve about the axis AC, then the point D will describe a circle, and the straight line DH will describe the surface of a cylinder, which will be always less than the surface described by DG, but will differ less from it the less that the length of DH is, and will ultimately be the limit of the surface described by DG; therefore, if p be $= 3{\cdot}1416$, DE $= y$, and AD $= v$, the fluxion of the surface will be $2py\dot{v}$, or if AE $= x$, then $\dot{v} = \sqrt{\dot{x}^2 + \dot{y}^2}$, and the fluxion will be $2py\sqrt{\dot{x}^2+\dot{y}^2}$, and the fluent of this derived from the nature of the curve will be the surface.

In the cylinder y is constant, and the fluent is $2pyv$ where v is the length of the cylinder.

EXAMPLES.

1. To find the surface of a cone. Here ADB is a straight line $= a$, Bc $= b$, and $a : b :: v : y = \frac{bv}{a}$. Therefore $2py\dot{v} = \frac{2pbv\dot{v}}{a}$, and the surface ADE $= \frac{pbv^2}{a}$, and the surface of the whole cone, ABC, where $v = a$ becomes $pba = 3{\cdot}1416 \times$ BC $\times$ AB, as in Prob. 7, Mensuration of Solids.

2. To find the surface of a sphere, where ADB is a circle, of which the radius AC $= a$. By Prop. 75, $\dot{v} = \frac{a\dot{x}}{y}$, and $2py\dot{v} = 2ap\dot{x}$; therefore the surface of the segment ADE $= 2pax = 3{\cdot}1416 \times 2$AC $\times$ AE, and the whole surface where AE becomes $= 2$AC $= 3{\cdot}1416 \times (2\text{AC})^2$, as in Prob. 13, Mensuration of Solids.

3. To find the surface of a parabolic conoid. Let $a =$ parameter, then $ax = y^2$ (Prop. 54), and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$, whence

$\dot{v} = \frac{\dot{y}}{a}\sqrt{a^2+4y^2}$, and $2py\dot{v} = \frac{2py\dot{y}}{a}\sqrt{a^2+4y^2}$, wherefore the corrected fluent is $\frac{p}{6a}\times(a^2+4y^2)^{\frac{3}{2}} - \frac{1}{6}pa^2$, the surface generated by AD.

4. To find the surface of a spheroid. Let $2a =$ fixed axis, $2b =$ revolving axis, $y =$ ordinate, and $x =$ distance of the ordinate from the centre, then $y = \frac{b}{a}\sqrt{a^2-x^2}$, and $\dot{y} = \frac{-b x\dot{x}}{a\sqrt{a^2-x^2}}$, and $\dot{v} = \frac{\dot{x}\sqrt{a^4-x^2\times(a^2-b^2)}}{a\sqrt{a^2-x^2}}$; therefore $2py\dot{v} = \frac{2pb\dot{x}}{a^2}\times\sqrt{a^4-x^2\times(a^2-b^2)}, = \frac{2pb\dot{x}}{a^2}\sqrt{a^4\mp d^2x^2}$ (putting $d^2 = a^2 - b^2$), the upper sign belongs to the oblong spheroid, where $a > b$, and the under sign to the oblate spheroid, where $a < b$. Suppose $\mathrm{P} =$ arc, of which the sine is $\frac{dx}{a^2}$, or $= \cdot 017453 \times$ degrees in that arc (radius $= 1$), when $a > b$, or let $\mathrm{P} =$ hyp. log. of $\frac{dx+\sqrt{a^4+d^2x^2}}{a^2}$, when $a < b$, and the surface will be $= \frac{pbx}{a^2}\sqrt{a^4\mp d^2x^2} + \frac{pba^2\mathrm{P}}{d}$. And for the hemispheroid where $x = a$, the arc is to be taken, of which the sine is $\frac{d}{a}$, or the log. of $\frac{b+d}{a}$, and the surface of the hemisphere will be $\frac{2pb}{d}\times(a^2\mathrm{P}+bd)$.

5. To find the surface of a hyperboloid. Let $2a =$ transverse axis, and $2b$ the conjugate, $y =$ ordinate, and $x =$ its distance from the centre, then $y = \frac{b}{a}\sqrt{x^2-a^2}$ and $\dot{y} = \frac{bx\dot{x}}{a\sqrt{x^2-a^2}}$; therefore $\dot{v} = \frac{\dot{x}\sqrt{d^2x^2-a^4}}{a\sqrt{x^2-a^2}}$ (putting $d^2 = a^2 + b^2$), and $2py\dot{v} = \frac{2pb\dot{x}}{a^2}\sqrt{d^2x^2-a^4}$, and therefore the surface will be $\frac{pbx}{a^2}\sqrt{d^2x^2-a^4} - \frac{pba^2}{d}\times$ hyp. log. of $dx+$

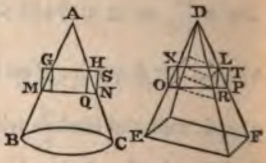
$\sqrt{d^2x^2-a^4}$, and the correction is $-pb^2+\log. a\times(b+d)$; therefore the whole surface will be $\frac{pbx}{a^2}\sqrt{d^2x^2-a^4}-pb^2$

$$+\frac{pba^2}{d}\times\log.\frac{a\times(b+d)}{dx+\sqrt{d^2x^2-a^4}}.$$

Section VI.

OF SOLIDS.

Prop. LXXX. Theorem. If two solids, ABC, DEF have the same height, and if their sections, at equal altitudes, by planes parallel to the bases, have always the same ratio which the bases have to one another, the solids have to one another the same ratio which their bases have.

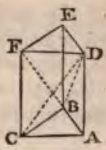


Let the section GH be at the same height with KL, and MN with OP. Upon their planes make the prisms or cylinders GQ, MS, and XR, OT. These solids have the same altitude, and therefore GQ : XR : : base GH : XL; that is, : : base BC : EF. For the same reason, MS : OT : : base BC : BF. In the same way it may be proved, that any series of prisms inscribed in ABC, is to a like series in DEF, as the base BC to EF, and the same of the circumscribed prisms. But the inscribed series may be taken of so small altitudes, that they will differ from the circumscribed by less than any given magnitude. The ratio of the prisms is therefore the ratio of the solids (see Limits of Ratios, Prop. 69.) Therefore the solids are to one another as their bases.

Cor. 1. If two pyramids or two cones be upon equal bases and of the same altitude, they are equal.

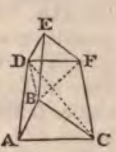
Cor. 2. A cone is equal to a pyramid of equal base and altitude with it.

Prop. LXXXI. Theorem. Every triangular prism ABC — DEF may be divided into three equal triangular pyramids.



Join FB, BD, DC; and because the triangle ADC = FDC, the pyramid ADC — B = FDC — B, but because the triangle EBF = FBC, the pyramid EBF — D = FBC — D, or FDC — B; therefore the prism ABC — DEF is divided into three equal pyramids ADC — B, FDC — B, and DEF — B.

Cor. 1. Hence a pyramid is the third part of a prism of equal base and altitude with it.

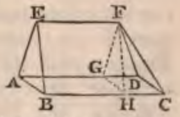


Cor. 2. The frustum of a triangular pyramid may be divided into three triangular pyramids, which are in continued proportion. For ADC — B : FDC — B : : ADC : FDC : : AC : DF; that is, : : BC : EF : : BCF : BEF, or : : BCF — D = FDC — B : BEF — D.

Cor. 3. The frustum of a pyramid is equal to two pyramids upon its two bases, and a pyramid of which the base is a mean proportional between the bases of the frustum, and all of the same altitude with the frustum.

Cor. 4. If A and a be similar sides of the bases, and A^2p the area of the one, a^2p will be the area of the other, and Aap the area of the mean; and if h be the height, the content of the frustum will be $(A^2+Aa+a^2)\,ph = ((A+a)^2 - Aa)ph$.

Prop. LXXXII. Theorem. A wedge ABCD — EF, of which the edge EF is equal to the length AD of the base is a triangular prism, and if the edge and length be unequal, the difference between the wedge and the prism is a pyramid DGHC — F, of which the base is a parallelogram, and the altitude is the perpendicular from the edge upon the base.



Cor. 1. Hence, if $AB = a$, $EF = BC = b$, and $CH = d$, and the perpendicular from E upon the base $= p$, the wedge or prism ABCD — EF $= a \times \frac{1}{2} bp$, and the pyramid CDGH — F $= a \times \frac{1}{3} dp$, and therefore the wedge ABHG — EF $= ap \times (\frac{1}{2}b \mp \frac{1}{3}d) = \frac{1}{6}ap \times (3b \pm 2d) = \frac{1}{6}ap \times (b + 2 \times (b \mp d))$, which is the rule in Prob. 11, Mensuration of Solids.

Cor. 2. The prismoid in Ex. 2. Prob. 12, Mensuration of Solids, may be divided into two wedges, by joining AH and BG, and making $EF = a$, $EH = b$, $AB = m$, $AD = n$, $a + m = p$, and $b + n = q$, then p and q are double the sides

of the middle base. The under wedge is $= (m + 2a)\, b \times \frac{1}{6} h$ ($h =$ height), and the upper wedge $= (a + 2m)\, n \times \frac{1}{6} h$; that is, they are together $= ((p + a)\, b + (p + m)\, n) \times \frac{1}{6} h - (p \times (b + n) + ab + mn) \times \frac{1}{6} h = (pq + ab + mn) \times \frac{1}{6} h$, which is the rule in Prob. 12, Mensuration of Solids.

Prop. LXXXIII. Theorem. A sphere or a spheroid is two-thirds of its circumscribing cylinder.

Let ABC be a semicircle or a semi-ellipse, AC the axis, OB perpendicular to AC, describe the parallelogram ADPC, join DO. Draw EF, GH parallel to OB, and let EF meet the circumference in L, and OD in K, and complete the rectangles GMKF, and GNLF. If the figure revolve about AC, the semicircle or semi-ellipse ABC will describe a sphere or a spheroid, ADPC a cylinder, ADO a cone. Also the figures GE, GL, and GK, will describe cylinders. Now, in the ellipse $AF \times FC : FL^2 :: AO^2 : OB^2 = AD^2 :: OF^2 : FK^2$; therefore $AF \times FC + OF^2 : FL^2 + FK^2 :: AO^2 : AD^2$, and $AF \times FC + OF^2 = AO^2$; therefore $FL^2 + FK^2 = AD^2 = EF^2$, and in the circle $AF \times FC = FL^2$, and $FK^2 = FO^2$; therefore $FL^2 + FK^2 = AO^2 = EF^2$; therefore the cylinder described by GL and GK, are together = cylinder described by GE. In the same manner, every cylinder in the hemisphere or hemispheroid, with the corresponding cylinder about the cone, is equal to the corresponding part of the cylinder described by AB, and the number of these cylinders may be increased, so that altogether they will not differ from the hemisphere and cone; therefore the hemisphere and cone are, together, equal to the circumscribing cylinder, and the cone is $\frac{1}{3}$ of the cylinder; therefore the sphere or spheroid is $\frac{2}{3}$ of its circumscribing cylinder.

Cor. 1. Hence any part of the sphere or spheroid, with the corresponding part of the cone, is equal to the corresponding part of the cylinder. Thus the segment described by ALF, together with the frustum described by ADKF, is equal to the cylinder described by ADEF. Let $AC = a$, $AF = h$, $FL = c$, and $FO = \frac{1}{2} a - h = FK$. Then in the sphere, the cylinder described by $FD = a^2 hp$ ($p = \cdot 7854$), and the conical frustum described by $ADKF = (3a^2 - 6ah + 4h^2) \times \frac{1}{3} hp$, and taking their difference, we have the segment described by $ALF = (3a - 2h) \times \frac{2}{3} h^2 p$, which is the rule in Prob. 15, Case 1, Mensuration of Solids.

And because $(a - h)\, h = c^2$; therefore $3a - 2h = \frac{3c^2 + h^2}{h}$.

By substituting this expression, the segment becomes $(3c^2+h^2) \times \frac{2}{3}ph$, which is the rule given in Prob. 15, Case 2, Mensuration of Solids.

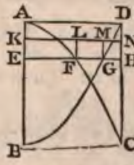
Again, the zone described by OFLB, together with the cone described by OFK, is equal to the cylinder described by OE; therefore making $OF = FK = m$, the cylinder $= a^2mp$, and the cone $= \frac{1}{3}m^2 \times mp$; therefore the zone described by $OFLB = (a^2 - \frac{1}{3}m^2)mp$, or if $a^2 - m^2 = FL^2 = d^2$, the zone is $(2a^2+d^2)\frac{1}{3}mp$, which is the rule for the middle zone in Prob. 16, Mensuration of Solids.

Again, from the zone described by $OFLB = (r^2+\frac{2}{3}h^2) \times ph$, (where $r = FL$, $h = OF$, and $p = 3{\cdot}1416$,) subtract the zone described by $OGNB = (R^2+\frac{2}{3}H^2) \times pH$, (where $R = GN$ and $H = OG$), the remainder will be the zone described by GFNL, which, when reduced by putting $m = FG = h - H$, and considering that $r^2+h^2 = R^2+H^2$, will become $(3R^2+3r^2+m^2) \times \frac{1}{6}mp$, which is the rule in Prob. 17, Mensuration of Solids.

Cor. 2. The sphere and its portions are to the spheroid and its corresponding portions as AO^2 to OB^2, from which consideration the rules in Prob. 18 and 19, Mensuration of Solids, are manifest.

Cor. 3. The sphere may be considered as a cone, of which the base is the surface of the sphere, and its vertex the centre; therefore, putting $S =$ surface, the sphere is $= \frac{1}{3}rS$, but the sphere is $=$ a cone upon one of its great circles, of which the height is $4r$, and is therefore $= \frac{4}{3}r \times r^2p$, $(p = 3{\cdot}1416)$; so that $\frac{4}{3}r \times r^2p = \frac{1}{3}rS$; therefore $S = 4r^2p = 4$ times the area of one of its great circles, which is the rule in Prob. 13, Mensuration of Solids.

Prop. LXXXIV. Theorem. A parabolic conoid is one-half of its circumscribing cylinder.

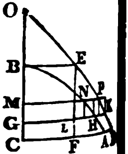


Let BAC and ABD be two equal parabolas, which have their vertices at A and B, and AB their common axis. Complete the rectangle ABCD, and draw EH, KN parallel to BC, and complete the rectangles EFLK, and EGMK, and let the whole revolve about the axis AB. By the property of the parabola (52.) $EF^2 : EG^2 :: AE : EB$, and $EF^2 : EF^2+EG^2 :: AE : AB :: EF^2 : BC^2 = EH^2$; therefore $EF^2+EG^2 = EH^2$, and therefore the cylinders described by EL and EM are, together, equal to the cylinder described by EN. And thus one

of the paraboloids with cylinders, which, together, are greater than the other paraboloid, is greater than the cylinder described by BD, and with cylinders less than the paraboloid, it is less than that cylinder; therefore the two paraboloids are equal to the cylinder, or the paraboloid is half the cylinder.

Cor. The paraboloid described by BEG, with the frustum described by BEFC, is equal to the cylinder described by BD. If, therefore, BC $= y$, BE $= x$, and EF $= z$, then EG$^2 = y^2 - z^2$, and the conoid described by BEG $= (y^2 - z^2)\frac{1}{2}px$, and the cylinder $= y^2 \times px$; therefore the frustum described by BEFC $= \frac{1}{2}(y^2 + z^2)px$. This is the rule in Prob. Mensuration of Solids.

Prop. LXXXV. The hyperbolic conoid is equal to the difference between the corresponding frustum of the asymptotic cone, and the cylinder of the same altitude, which has the conjugate axis for the diameter of its base.



Let BCA be a hyperbola, of which OBC is the transverse axis, and OD the asymptote, draw the tangent BE, it is = the conjugate semi-axis. Draw any two straight lines GK, MP, parallel to CD, and complete the rectangles MH, MK, GN, GP, and CE, and let the whole revolve about BC. Because GH^2 + BE^2 = GK^2, the cylinders described by the rectangles ML and MH are equal to that described by MK. And for the same reason, the cylinders described by GN and ML are equal to that described by GP. Therefore the cylinder described by CE, together with any series of cylinders about the hyperboloid, is greater than the frustum described by BEDC, and with any series in the hyperboloid, it is less than the frustum; therefore the cylinder and hyperboloid are equal to the frustum.

Cor. 1. If OB $= a$, BE $= c$, BC $= x$, and CA $= y$, then CD $= \frac{c}{a}(a + x)$. And the conic frustum made by BCDE $= (a^2 + ax + \frac{1}{3}x^2)\frac{c^2xp}{a^2}$, and the cylinder made by CE $= \frac{a^2c^2xp}{a^2}$, and taking their difference, the hyperboloid $= (ax + \frac{1}{3}x^2) \times \frac{c^2xp}{a^2}$, or putting $\frac{y^2}{2ax + x^2}$ instead of $\frac{c^2}{a^2}$, it be

comes $\frac{ax+\frac{1}{3}x^2}{2ax+x^2}\times y^2xp=\frac{2a+\frac{2}{3}x}{2a+x}\times\frac{1}{2}y^2xp$, which is the rule in Prob. 23, Mensuration of Solids.

Cor. 2. If $CG=x$, and $BG=m$, the content of the hyperboloid described by CBA, will be $\frac{c^2p}{a^2}\times(a\times\overline{m+x}|^2+\frac{1}{3}\overline{m+x}|^3)$, and the content described by GBH will be $\frac{c^2p}{a^2}\times(am^2+\frac{1}{3}m^3)$ and their difference $=\frac{1}{2}\frac{c^2px}{a^2}\times(4am+2ax+2m^2+2mx+\frac{2}{3}x^2)$ will be the content of the frustum described by CGHA. But $\frac{1}{2}pxy^2=\frac{\frac{1}{2}c^2px}{a^2}(2am+2ax+m^2+2mx+x^2)$, and putting $GH=v$, $\frac{1}{2}pxv^2=\frac{\frac{1}{2}c^2px}{a^2}\times(2am+m^2)$, and the sum of these two $\frac{1}{2}px\times(y^2+v^2)=(4am+2ax+2m^2+2mx+x^2)\times\frac{\frac{1}{2}c^2px}{a^2}$, which exceeds the content by $\frac{\frac{1}{2}c^2px}{a^2}\times\frac{1}{3}x^2$, wherefore the content of the frustum is $=\frac{1}{2}px\left(y^2+v^2-\frac{c^2x^2}{3a^2}\right)$, which is the rule in Prob. 24, Mensuration of Solids.

Prop. LXXXVI. Problem. To find the content of a circular spindle, described by the revolution of the segment ABC about its chord AC.

Let $BO=r$, $OE=d$, $AE=c$, $EH=x$, and $HN=y$, then $c^2=r^2-d^2$, and $(d+y)^2=r^2-x^2$, whence $y^2=r^2-x^2-d^2-2dy=c^2-x^2-2dy$. Now by what was shown in (65.), the fluxion of the solidity is $=\dot{x}\times$ circle described by $NH=py^2\dot{x}=pc^2\dot{x}-px^2\dot{x}-2pdy\dot{x}$, and $y\dot{x}$ is the fluxion of the area BEHN; therefore, taking the fluent, the content of the zone described by $BEHN=p\times(c^2x-\frac{1}{3}x^3-2d\times BEHN)$. This is the rule for the zone, Prob. 26, Mensuration of Solids.

And when x becomes $=c$, the content of half the spindle will be $2p\times(\frac{1}{3}c^3-d\times ABE)$, which is the rule for the spindle in Prob. 25, Mensuration of Solids.

Cor. 1. If ABC be a segment of an ellipse, and $a =$ semi-axis parallel to AC, y^2 will be found to be $= \frac{r^2}{a^2}(c^2 - x^2) - 2dy$, and $py^2\dot{x} = \frac{pr^2}{a^2} \times (c^2\dot{x} - x^2\dot{x}) - 2pdy\dot{x}$, where as before $y\dot{x} =$ fluxion of BEHN; therefore the content of the zone described by BEHN $= \frac{pr^2}{a^2} \times (c^2x - \frac{1}{3}x^3) - 2pd \times$ BEHN, which is the rule for the zone in Prob. 28, Mensuration of Solids. And when $x = c$, the content of half the spindle is $2p \times \left(\frac{\frac{1}{3}r^2c^3}{a^2} - d \times \text{ABE}\right)$.

If $r - d = m$ and $S =$ area ABE, the half spindle $= \frac{2}{3}pc \times \left[m^2 - d\left(\frac{3S}{c} - m\right)\right]$, which is the rule in Prob. 27, Mensuration of Solids.

Cor. 2. If the frustum be taken from half the spindle, there will remain the segment described by the revolution of AHN about AH, and if $AH = h$, it will be in the circle $= p \times (\frac{1}{3}h^2 \times (3c - h) - 2d \times \text{AHN})$. And in the ellipse $= p \times \left(\frac{r^2h^2}{3a^2} \times (3c - h) - 2d \times \text{AHN}\right.$.

Prop. LXXXVII. Problem. To find the content of a parabolic spindle, described by the revolution of the parabola ADC, about its ordinate AC.

Let $AE = a$, $ED = c$, $AG = x$, and $GH = y$, then by the property of the parabola $c : c - y :: a^2 : \overline{a - x}|^2$; therefore $c - y = \frac{c \times \overline{a - x}|^2}{a^2}$ and $y = c \times \frac{2ax - x^2}{a^2}$; therefore $py^2\dot{x} = \frac{pc^2x^2\dot{x}}{a^4} \times 2\overline{a - x}|^2 =$ $\frac{4pc^2a^2x^2\dot{x}}{a^4} - \frac{4pc^2ax^3\dot{x}}{a^4} + \frac{pc^2x^4\dot{x}}{a^4}$, and the fluent or value of the segment described by AHG is $= \frac{4pc^2a^2x^3}{3a^4} - \frac{pc^2ax^4}{a^4} + \frac{pc^2x^5}{5a^4}$. And when $x = a$, the half spindle described by

$\text{AED} = \frac{4pc^2a^5}{3a^4} - \frac{pc^2a^5}{a^4} + \frac{pc^2a^5}{5a^4} = \frac{8pc^2a}{15} = \frac{8}{15}$ of the circumscribing cylinder, which is the rule in Prob. 29, Mensuration of Solids.

Cor. If the segment described by AGH be taken from half the spindle, there will remain the zone or frustum described by $\text{DEGH} = pc^2 \times \left(\frac{8a}{15} - \frac{4x^3}{3a^2} + \frac{x^4}{a^3} - \frac{x^5}{5a^4}\right)$ or by substituting for $a-x$ its equal $a\sqrt{\frac{c-y}{c}}$, or $x = a - a\sqrt{\frac{c-y}{c}}$, it becomes $p \times \overline{a-x} \times \frac{8c^2+4cy+3y^2}{15} = \frac{1}{3}p \times \overline{a-x} \times (2c^2 + y^2 - \frac{2}{5} \times \overline{c-y}|^2)$, which is the rule in Prob. 30, Mensuration of Solids.

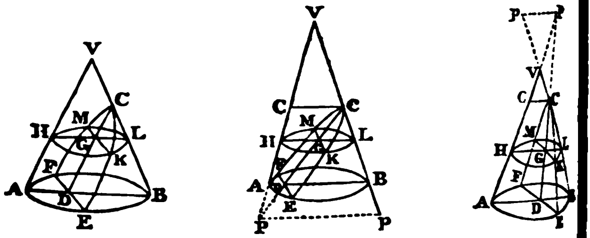
Prop. LXXXVIII. Problem. To find the content of the hoof of a cylinder ABC–FHG, cut off by the plane DFB.

Suppose the hoof to be generated by the triangle ECF, moving parallel to itself along BD. Let $FC = h$, $CE = v$, $EB = s$, $AC = 2r$, cosine $CB = c = r - v$ or $v - r$. The area of the segment $DCB = A$. Let $x =$ distance of the moving triangle from $AC =$ sine of the arc between it and C, and let $y =$ cosine of the same arc. Then $y - c =$ base of the moving triangle, and $v : h :: y - c :$ its height $= \frac{h}{v} \times \overline{y-c}$; therefore the area of the moving triangle is $\frac{h}{2v}(y-c)^2$, and the fluxion of the hoof will be $\frac{h\dot{x}}{2v}(y-c)^2$, but $(y-c)^2 = y^2 - c^2 - 2c(y-c) = s^2 - x^2 - 2c(y-c)$; therefore the fluxion becomes $\frac{h\dot{x}}{2v} \times (s^2 - x^2 - 2c(y-c))$, and $\frac{ch\dot{x}}{v}(y-c)$ is $= \frac{ch}{v} \times$ the fluxion of the area generated by the base of the triangle between that base and CE. Let this area be called B, and the

content will be $\frac{hx}{2v}(s^2 - \frac{1}{3}x^2) - \frac{hcB}{v}$, and when $x = s$, the half-hoof becomes $\frac{h}{2v} \times (\frac{2}{3}s^3 - cA)$.

Cor. If E coincide with the centre O, then $c = 0$, and the hoof becomes $\frac{2}{3}r^2h$.

Prop. LXXXIX. Theorem. If a cone be cut by a plane which neither passes through the vertex nor is parallel to the base, the section made by it will be a conic section.



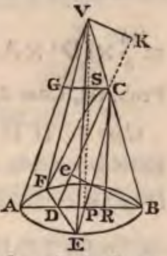
Let the cone AEB–V, of which the base is the circle AEBF and vertex V, be cut by a plane, which forms the section ECF, this is a conic section. Let it meet the base in the line EDF, and draw the diameter AB perpendicular to EF, and join AV and BV, and let the plane ABV cut the section ECF in CD. Let a plane parallel to the base cut the cone in the circle HKL, and the planes ABV and ECF in HL and KGM. The base AEB is perpendicular to AVB; therefore DE, and the plane ECF are perpendicular to AVB, and the angles EDC, KGC are right angles. And because AVB bisects the cone, ED = DF and KG = GM, and the rectangle $AD \times DB = DE^2$ and $HG \times GL = GK^2$ (by Prop. 17.)

First, Let CD be parallel to AV, then AD = HG and CD : CG : : DB : GL : : $AD \times DB : HG \times GL$: : $DE^2 : GK^2$, which is the property of the parabola.

Next, Let CD meet AV in P. Then PD : PG : : AD : GH and CD : CG : : DB : GL; therefore $PD \times DC : PG \times GC$: : $AD \times DB : HG \times GL$: : DE : GK^2, which is the property of the ellipse or hyperbola, (53.) viz. of the ellipse, if P be below V, and of the hyperbola, if P be above V.

Cor. In the ellipse and hyperbola. If Cc, Pp be parallel to AB, then $\sqrt{Cc \times Pp}$ = conjugate axis.

PROP. XC. Problem. To find the content of the hoof EBF–C of a cone AEB–V, cut off from it by the oblique plane ECF.



Draw CR, VP perpendicular to AB and VK, Be perpendicular to CD, and CG parallel to AB. As BR : BP : : CR : PV, and BR : CS = RP : : CR : VS, and because CR : VS : : BC : CV : : Be : VK; therefore VS : VK : : CR : Be : : CD : DB. Join EV, FV. The solid EBF–V is pyramidal or conical of which the base is EBF, and its height VP; it is therefore $= \frac{1}{3}$ VP $\times$ EBF. And the solid ECF–V has EFC for its base, and VK for its altitude; it is therefore $= \frac{1}{3}$ VK $\times$ ECF. Wherefore the hoof EBF–C, which is the difference of these solids, is $= \frac{1}{3}$ VP $\times$ EBF $- \frac{1}{3}$ VK $\times$ ECF.

Let AB = D, CG = d, DB = v, DC = m, CR = h, and let $a = \mathrm{D} - d$, then $\mathrm{PV} = \frac{\mathrm{D}h}{a}$, $\mathrm{VS} = \frac{dh}{a}$, $\mathrm{VK} = \frac{vdh}{am}$, and if A be the tabular area of the segment similar to EBF, the diameter = 1, and the versed sine $= \frac{v}{\mathrm{D}}$, then $\mathrm{D}^2 \times \mathrm{A} = \mathrm{EBF}$. And these values being substituted, the hoof becomes $\frac{\frac{1}{3}h}{a} \times (\mathrm{D}^3 \times \mathrm{A} - \frac{vd}{m} \times \mathrm{ECF}.)$

Case 1. If DC be parallel to AV, or AD = CG, the base ECF is a parabola, and its area is $= \frac{2}{3}$ EF $\times$ CD $= \frac{2}{3}$ CD $\times 2\sqrt{\mathrm{AD} \times \mathrm{DB}}$, and if this be substituted, the hoof becomes $\frac{\frac{1}{3}h}{a} \times (\mathrm{D}^3 \times \mathrm{A} - \frac{4}{3} vd \sqrt{dv})$, or because $v = a = \mathrm{D} - d$, the hoof is $\frac{1}{3} h \times \left(\frac{\mathrm{D}^3 \times \mathrm{A}}{a} - \frac{4d}{3}\sqrt{ad}.\right)$

Case 2. If DC meet AV, or if ECF be a segment of an ellipse, then $v > a$; the whole axis is $= \frac{md}{v - a}$, and its conjugate $= d\sqrt{\frac{v}{v-a}}$. And if B be the tabular segment of which

the diameter is 1, and the versed sine $m \div \frac{md}{v-a} = \frac{v-a}{d}$, then the area ECF $= \frac{md^2 v^{\frac{1}{2}}}{\overline{v-a}|^{\frac{3}{2}}}$B. And therefore the hoof EBFC $= \frac{\frac{1}{3}h}{a} \times (\mathrm{D}^3 \times \mathrm{A} - d^3 \times \left(\frac{v}{v-a}\right)^{\frac{3}{2}}\mathrm{B}.)$ This is the rule in Prob. 33, Case 2; Mensuration of Solids.

Case 3. If D coincide with A, then $v = \mathrm{D}$, the segment EBF is a circle, and ECF an ellipse, the area of the circle is $\mathrm{D}^2 p$ ($p = \cdot 7854$), and of the ellipse $pm\sqrt{\mathrm{D}d}$, and therefore the hoof is $\frac{1}{3}hp\mathrm{D} \times \frac{\mathrm{D}^2 - d\sqrt{\mathrm{D}d}}{\mathrm{D}-d}$. And the other hoof ACG $= \frac{1}{3}hp\mathrm{D} \times \frac{\mathrm{D}\sqrt{\mathrm{D}d} - d^2}{\mathrm{D}-d}$.

Case 4. If the segment ECF be a hyperbola, the transverse is $\frac{md}{a-v}$, and the conjugate $d\sqrt{\frac{v}{a-v}}$, and FG $= 2\sqrt{\overline{\mathrm{D}-v} \times v}$, and the area may be found by Prob. 29, Mensuration of Surfaces, and if it be called B, the hoof will be $\frac{\frac{1}{3}h}{a} \times (\mathrm{D}^3 \times \mathrm{A} - \frac{vd}{m}\mathrm{B})$.

Case 5. If CD be perpendicular to AB, or coincide with CR, then $v = \frac{1}{2}(\mathrm{D} - d)$, and $m = h$, and the transverse $= \frac{2dh}{a}$, and the conjugate $= d$, and the hoof becomes $\frac{\frac{1}{3}h\mathrm{D}^3 \times \mathrm{A}}{a} - \frac{1}{6}d\mathrm{B}$.

SPHERICAL TRIGONOMETRY.

DEFINITIONS AND PRINCIPLES.

ERICAL TRIGONOMETRY is that branch of Mathematics ich shows how to compute the sides and angles of spherical angles.

A SPHERE is a solid bounded by a curve surface, every nt of which is equally distant from a point within it, led the *centre*.

A sphere may be conceived to be generated by a semicircle olving about its diameter.

The *axis* or *diameter* of a sphere is a straight line passing ough the centre, and both ends terminating at the surface.

Any circle formed from the section of a sphere by a plane sing through its centre, is called a *great circle* of the ere; and all others *small circles*.

The *pole* of a great circle is a point on the surface of the ere, equally distant from every point in the circumference hat circle.

A *spherical angle* is the angle made by two arcs of great cles, and is the same with the inclination of the planes of se circles, or with the plane angle made by the tangents to se arcs at the point of intersection.

A *spherical triangle* is a figure formed upon the surface of here by the intersection of the arcs of three great circles.

A spherical triangle is called RIGHT-ANGLED, when it has right angle; QUADRANTAL, when it has one side equal to ; and OBLIQUE-ANGLED, when it has none of its angles right les. It is also called *equilateral*, when the three sides are al; *isosceles*, when two sides are equal; and *scalene*, when the three sides are unequal.

NOTE. A right-angled spherical triangle may have *one*, , or *three* right angles, and in the last case it is likewise drantal, and the angles and sides are known.

Arcs or angles are said to be *alike*, or of the *same af- tion*, when both are less or both greater than a qua- nt; and they are said to be *unlike*, or of *different ection*, when the one is greater and the other less than a adrant.

PROP. I. If a sphere be cut by a plane in any direction, the section will be a circle.

PROP. II. The arc of a great circle, between the pole and the circumference of another great circle, is a quadrant.

Cor. 1. The straight line drawn from the pole of any great circle to the centre of the sphere, is at right angles to the plane of that circle; and conversely.

Cor. 2. The poles of a great circle are the extremities of the axis of the sphere, which is perpendicular to the plane of that great circle.

PROP. III. A spherical angle at the pole of a great circle is measured by the arc of that great circle, intercepted between the circles which contain the angle.

PROP. IV. If two arcs of different great circles be drawn from the same point, and each of them be a quadrant, that point is the pole of the great circle which passes through the extremities of these arcs.

Cor. 1. A great circle drawn through the pole of another great circle cuts it at right angles.

Cor. 2. Great circles, whose planes are perpendicular to the plane of one and the same great circle, meet in the poles of that circle.

PROP. V. If two spherical triangles have the three sides of the one equal to the three sides of the other, each to each, the angles which are opposite to the equal sides are likewise equal; and conversely.

PROP. VI. If two sides and the included angle of one spherical triangle be equal to two sides and the included angle in another, these two triangles are equal in every respect.

PROP. VII. The angles at the base of an isosceles spherical triangle are equal to one another.

Cor. 1. If two of the angles of a spherical triangle be equal to one another, the sides opposite to them are also equal.

Cor. 2. If a perpendicular be drawn from the vertex of an isosceles spherical triangle to the base, it will bisect both the vertical angle and the base, except when the two sides are quadrants, in which case the number of perpendiculars is indefinite.

PROP. VIII. Any two sides of a spherical triangle are together greater than the third side, and the difference of any two sides is less than the third.

Cor. The arc which passes through any two points on the surface of a sphere is the shortest distance between these points.

PROP. IX. The three sides of a spherical triangle are together less than the circumference of a great circle; and the difference of any two sides is less than half the circumference.

PROP. X. The greater angle of a spherical triangle has the greater side opposite to it; and conversely.

PROP. XI. If two sides of a spherical triangle be together equal to, greater, or less, than a semicircle, the sum of their opposite angles will be equal to, greater, or less, than two right angles; and conversely.

Cor. 1. If each side of a spherical triangle be equal to, greater, or less, than a quadrant, each of the angles will, accordingly, be right, obtuse, or acute; and conversely.

Cor. 2. Half the sum of any two sides of a spherical triangle is of the same affection as half the sum of their opposite angles.

PROP. XII. If from the angular points of a spherical triangle as poles there be described on the surface of the sphere three arcs of great circles, which by their intersection form another spherical triangle, each side of this new triangle will be the supplement of the measure of the angle which is at its pole; and the measure of each of its angles will be the supplement to that side of the primitive triangle to which it is opposite.

Cor. Hence these two triangles are called supplemental or *polar* triangles.

PROP. XIII. The three angles of a spherical triangle are together greater than two and less than six right angles.

Cor. 1. The three angles, together with twice the supplement of the least, are less than six right angles.

Cor. 2. The sum of any two angles is greater than the supplement of the third angle.

PROP. XIV. In any right-angled spherical triangle

the sides about the right angle are of the same affection with their opposite angles; and conversely.

Cor. The same is also the case in any quadrantal triangle.

Prop. XV. In any right-angled spherical triangle the hypotenuse is greater or less than a quadrant, according as the two sides about the right angle are of the same or of different affection; and conversely. If one of the sides be a quadrant, the hypotenuse is also a quadrant.

Cor. The hypotenuse will be greater or less than a quadrant, according as the angles are of the same or of different affection, because the angles are of the same affection as their opposite sides.

Prop. XVI. In any spherical triangle, if the perpendicular drawn from the vertex to the base fall within the triangle, the angles at the base are of the same affection; and if it fall without the triangle, they are of different affection; and conversely.

STEREOGRAPHIC PROJECTION OF THE SPHERE.

DEFINITIONS AND PRINCIPLES.

To project an object is to represent every point of it upon the same plane as it appears to the eye in a certain position.

The *plane* of projection is that upon which the object is projected, and the point where the eye is situated is called the projecting point.

The *stereographic* projection is a representation of the circles of the sphere upon the plane of one of its great circles, such as they would appear to an observer placed in one of the poles of that circle.

The great circle, upon the plane of which the projection is made, is called the *primitive.*

By the *semitangent* of any arc is meant the tangent of half that arc.

The *line of measures* of any circle of the sphere is that diameter of the primitive produced indefinitely, which is perpendicular to the line of common section of the circle and the primitive.

The representation or projection of any point in the sphere is the point in which the straight line drawn from it to the projecting point intersects the plane of projection.

Prop. I. Every great circle of a sphere, which passes through the projecting point, is projected into a straight line passing through the centre of the primitive; and every arc of it, reckoned from the other pole of the primitive, is projected into its semitangent.

Cor. 1. Every small circle which passes through the projecting point is projected into that straight line which is its common section with the primitive.

Cor. 2. Every straight line in the plane of the primitive, and produced indefinitely, is the projection of some circle on the sphere passing through the projecting point.

Cor. 3. The stereographic projection of any point on the surface of the sphere is distant from the centre of the primitive by the semitangent of the distance of that point from the pole opposite the projecting point.

Prop. II. Every circle of the sphere, which does not pass through the projecting point, is projected into a circle.

Cor. 1. The centres and poles of all circles parallel to the primitive have their projections in its centre.

Cor. 2. The centres and poles of every circle inclined to the primitive have their projections in the line of measures.

Cor. 3. All projected great circles cut the primitive in two points diametrically opposite.

Prop. III. The centre of the projection of a great circle is distant from the centre of the primitive by the tangent of that great circle's inclination to the primitive, and its radius is the secant of the same.

Prop. IV. The centre of projection of a small circle perpendicular to the primitive is distant from the centre of the primitive by the secant of the distance of the circle from its nearest pole, and the radius of projection is the tangent of the same.

Prop. V. The projection of the poles of any great circle inclined to the primitive is in the line of measures distant from the centre of the primitive by the tangent and cotangent of half its inclination.

Prop. VI. Any two circles upon the sphere passing through the poles of two great circles, intercept equal arcs upon these circles.

Prop. VII. If from either pole of a projected great circle two straight lines be drawn to meet the primitive and the projection, they will intercept corresponding arcs of these circles.

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Every spherical triangle consists of six parts,—three sides and three angles,—any three of which being given, the rest may be found.

In a right-angled spherical triangle the right angle can never be the subject of inquiry; and therefore there are only the three sides and the two oblique angles presented to our consideration, and of these the two sides, containing the right angle and the *complements* of the angles and of the hypotenuse, are called the FIVE CIRCULAR PARTS.

When any one of these is taken as the MIDDLE PART, the two which are immediately adjacent to it on the right and left are called the ADJACENT PARTS; and the other two, each being separated from the middle part by an adjacent part, are called OPPOSITE PARTS.

With this arrangement of the different parts, the solution, in every case, is obtained by the two following equations.

1. Rad. $\times$ sin. middle part $=$ the rectangle of the tangents of the adjacent parts.

2. Rad. $\times$ sin. middle part $=$ the rectangle of the cosines of the opposite parts.

Note. In applying these equations to the solution of problems take that as the middle part which is either adjacent to the other two given parts, or is separated from them by the remaining parts of the triangle, and form the equations according as the remaining parts are adjacent or opposite.

These equations may be transformed into proportions having the required part for the last term from whence its value will be obtained.

A *quadrantal triangle* may be changed into a right-angled triangle, by calling the supplement of the angle opposite to the quadrantal side, the hypotenuse; the other angles, the sides; the quadrantal side, radius; and the other sides, angles; but in the solution we must substitute *same* for *different affection* in the limitation.

The following Table contains the proportions for the solution of the sixteen cases of any right-angled spherical triangle ABC (see figure, Case 1.).

Given.	Sought.	Solution.	Equations.	Limitation.	Cases.
BC & B	AC	R : sin. BC : : sin. B : sin. AC.	2	of the same affection with B.	1
	AB	R : cos. B : : tan. BC : tan. BA.	1	less than 90°, when BC and B are of the same affection.	2
	C	R : cos. BC : : tan. B : cot. C.	1	otherwise greater than 90°.	3
AC & C	AB	R : sin. AC : : tan. C : tan. AB.	1	of the same affection with C.	4
	BC	cos. C : R : : tan. AC : tan. BC.	1	less than 90°, when AC and C are of the same affection.	5
	B	R : sin. C : cos. AC : cos. B.	2	of the same affection with AC.	6
AC & B	AB	tan. B : tan. AC : : R : sin. AB.	1	ambiguous; for two triangles may have the given	7
	BC	sin. B : R : : sin. AC : sin. BC.	2	things, but have the things sought in one of them	8
	C	cos. AC : R : : cos. B : sin. C.	2	the supplements of the things sought in the other.	9
AC & CB	AB	cos. AC : R : : cos. BC : cos. BA.	2	less than 90°, if AC and CB be of the same affection.	10
	B	sin. BC : R : : sin. AC : sin. B.	2	of the same affection with AC.	11
	C	tan. CB : tan. CA : : R : cos. C.	1	less than 90°, if AC and CB be of the same affection.	12
AB & AC	BC	R : cos. AC : : cos. AB : cos. BC.	2	less than 90°, if AB and AC be of the same affection.	13
	B	sin. AB : R : : tan. AC : tan. B.	1	of the same affection with AC.	14
	C	sin. AC : R : : tan. AB : tan. C.	1	of the same affection with AB.	14
B & C	AB	sin. B : R : : cos. C : cos. AB.	2	of the same affection with C.	15
	AC	sin. C : R : : cos. B : cos. AC.	2	of the same affection with B.	15
	BC	tan. B : cot. C : : R : cos. BC.	1	less than 90°, if B and C be of the same affection.	16

CASE I. GIVEN THE HYPOTENUSE AND AN ANGLE.

1. In the right-angled spherical triangle ABC are given the hypotenuse BC 63° 30′, and the angle ABC 53° 42′; to find the sides AB, AC, and the angle ACB.

Construction. Draw the radius OF of the primitive BAD. Make OE the semi-tangent, and OF the tangent of 53° 42′, then E is the pole of the hypotenuse, and F its centre, from which, with the secant of 53° 42′, describe the circle BCD. From B to I lay 63° 30′ on the primitive, draw a straight line from its extremity I to E, cutting BCD in C, and draw the radius OCA; then ACB is the triangle. The side AB is measured on the line of chords. OC measured on the line of semi-tangents, and subtracted from 90°, or AC reckoned on the line of semi-tangents from 90° backward, gives the arc AC. Extend the straight line IE to H, and HD, measured on the line of chords, gives the angle ACB.

Calculation. The five parts of this triangle are BC, the angles at B and C, and the complements of AB and AC, which are AG and OC. Of these, BC and B are given; and of the things required, BA and C are adjacent to given things, and are therefore found by Equa. 1; and AC being separated from given things, is found by Equa. 2.

By Equa. 1. R : cos. BC : : tan. B : cot. C; and R : cos. B : : tan. BC : cot. AG = tan. AB. And by Equa. 2. R : sin. B : : sin. BC : cos. CO = sin. CA. And all the three are acute. For CA is of the same affection with B. And AB and C are acute, because BC and B are of the same affection.

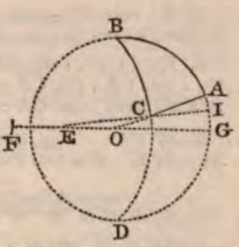
BC 63° 30′	cos. 9·6495274	tan. 10·3022637	sin. 9·9517912
B 53° 42′	tan. 10·1339650	cos. 9·7723314	sin. 9.9062964
C 58° 43′ 28″	cot. 9·7834924	AB tan. 10·0745951	CA sin. 9·8580876
		AB acute = 49° 53′ 48″	CA acute = 46° 9′ 29″

2. Given the hypotenuse BC 126° 24′, and the angle B 57° 22′; to find the rest. Ans. The angle C 132° 49′ 18″, the sides AB 36° 10′ 59″, and AC 137° 19′ 32″.

3. Given the hypotenuse BC 72° 28′, and the angle B 138° 23′; to find the rest. Ans. The angle C 104° 58′ 58″, the sides AB 112° 54′ 32″, and AC 140° 42′ 24″.

CASE II. GIVEN A SIDE AND THE ADJACENT ANGLE.

1. In the spherical triangle ABC, right-angled at A, are given the side AB 51° 28′, and the angle ABC 66° 48′; to find the hypotenuse BC, the side AC, and the angle at C.



Construction. Draw the diameter GF of the primitive ABD. Make OE the semi-tangent of 66° 48′, and OF its tangent. From F, with the secant of 66° 48′ for a radius, describe the circle BCD; make BA 51° 28′, and draw ACO, then ABC is the triangle.

AC, or its complement CO, is measured on the line of semi-tangents. Draw a line from E through C to I, and the distance of B from the point I, where it cuts BAG, gives BC; and the distance of D from its other extremity gives the angle at C.

Calculation. The hypotenuse BC, and the side CA, being adjacent to given things, are found by Equa. 1., and the angle C by Equa. 2.

Thus; 1. R : cos. AG = sin. AB : : tan. B : cot. OC = tan. CA, like B; and cos. B : R : : cot. AG = tan. AB : tan. BC, acute, for BA is like B. Also, 2. R : sin. B : : sin. AG = cos. AB : cos. C, like AB.

AB 51° 28′	sin.	9·8933433	R + tan. 20·0988763	cos. 9·7944670
B 66° 48′	tan.—R	0·3679473	cos. 9·5954322	sin. 9·9633795
AC 61° 16′ 52″	tan.	10·2612906	BC tan. 10·5034441	C cos. 9·7578465
			BC = 72° 34′ 54″	C = 55° 4′ 7″

2. Given the side AB 126° 26′, and the angle B 142° 48″; to find the rest. Ans. Hyp. BC 59° 32′ 45″, side AC 148° 35′ 17″, and the angle C 111° 2′ 34″.

3. Given the side AB 57° 44′, and the angle B 112° 26′; to find the rest. Ans. Hyp. BC 103° 32′ 46″, the side AC 116° 1′ 26″, and the angle C 60° 25′ 54″.

CASE III. GIVEN A SIDE AND THE OPPOSITE ANGLE.

1. In the spherical triangle ABC, right-angled at A, are given the side AC 38° 27′, and the opposite angle ABC 57° 48′; to find the hypotenuse BC, the side AB, and the angle at C.

Construction. On OA the radius of the primitive make OC 51° 33′ the complement of AC. With the tangent of 57° 48′ describe an arc from O, and with the secant of 57° 48′ from C cut that arc in F, from which centre describe the circle BCD, then either ABC or ADC is the triangle. AB is measured on the line of chords, and BC and C as in the last case.

Calculation. AB being adjacent to given things, is found by Equa. 1., and BC and C by Equa. 2. They are all ambiguous, or have two values.

1. R : cos. AG = sin. AB : : tan. B : cot. CO = tan. AC, and tan. B : tan. AC : : R : sin. AB or AD.

2. R : sin. B : : sin. BC : cos. CO = sin. CA, and (inver.) sin. B : R : : sin. CA : sin. CB or CD. R : sin. OC = cos. AC : : sin. C : cos. B, and (inver.) cos. CA : R : : cos. B : sin. ACB or ACD.

B=57° 48′	tan. 10.2008431	sin. 9.9274695	R+cos. 19.7266264
AC=38° 27′	R + tan. 19.8998271	R+sin. 19.7936727	cos. 9·8938456
	Sine of AB=9.6989840	Sin. BC 9.8662032	Sin. C 9.8327808

AB=30° 0′ 4″ or 149° 59′ 56″ BC=47° 17′ 43″ or 132° 42′ 17″

C=42° 52′ 37″ or 137° 7′ 23″

2. Given the side AC 136° 28′, and the angle B 127° 48′; to find the rest. Ans. The hyp. BC 60° 39′ 24″; the side AB 47° 28′ 20″; and the angle C 57° 43′ 1″, or their supplements.

3. Given the angle B 84° 21′, and the side AC 78° 40′; to find the rest. Ans. The hyp. BC 80° 9′ 34″; the side AB 29° 34′ 42″; and the angle C 30° 3′ 54″, or their supplements.

CASE IV. WHEN THE HYPOTENUSE AND A SIDE ARE GIVEN.

1. In the spherical triangle ABC, right-angled at A, are given the hypotenuse BC 64° 42′, and the side AC 47° 48′; to find the side AB, and the angles at B and C.

Construction. Lay AC 47° 48′ on the primitive, and draw the radii OC, OA. On the former lay the secant of 64° 42′ from O to H, from which, with the tangent of 64° 42′, cut OA in B, and describe the circle CBD, then ABC is the triangle.

Let F be the centre of CBD, then OF measured on the line of tangents gives the angle ACB. Lay the semi-tangent of it from O to E. Lay a ruler from B through E; the arc of the primitive between it and D is the measure of the angle at B, and OB measured on the line of semi-tangents gives the complement of AB.

Calculation. The angle at C being adjacent to the given things, is found by Equa. 1; the other two, being separated from them, are found by Equa. 2.

1. Tan. CB : cot. AG = tan. AC : : R : cos. C acute, since AC, CB are alike.

2. Sin. CB : R : : cos. AG = sin. AC : sin. B, like AC. Sin. AG=cos. AC : R : : cos. BC : sin. OB=cos. BA acute, because AC, CB are alike.

BC 64° 42′	R+cos. 19.6307917	sin. 9·9562081	tang. 10·3254164
AC 47° 48′	cos. 9·8271887	R+sin. 19·8697037	R+tang. 20·0425150
AB 50° 29′ 24″	cos. 9·8036030	sin. B. 9·9134956	cos. C 9·7170986

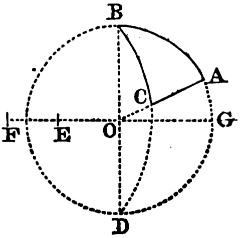
B = 55° 1′ 29″ and C = 58° 34′ 47″

2. Given the hypotenuse BC 121° 12′, and the side AC 56° 15′; to find the rest. Ans. The angles C 155 0′ 34″, and B 76° 25′ 31″; and the side AB 158° 48′ 57″.

3. Given the hypotenuse BC 72° 28′, and the side AC 123° 16′; to find the rest. Ans. The angles C 118° 47′ 19″, and B 118° 44′ 1″, and the side AB 123° 18′ 46″.

CASE V. GIVEN THE SIDES ABOUT THE RIGHT ANGLE.

1. Given the sides AB 47° 38′, and AC 67° 30′, about the right angle BAC of the spherical triangle ABC; to find the hypotenuse BC, and the angles at B and C.



Construction. Make AB 47° 38′ on the primitive, and draw the radius OA, on which make OC = 22° 30′, the complement of AC taken from the line of semi-tangents, and having drawn the diameter BD, describe the circle BCD; then ABC is the triangle.

Let F be the centre of BCD, then OF measured on the line of tangents gives the angle at B. Make OE its semi-tangent, then E is the pole of BCD, and BC and C are measured as in the 2d Case.

Calculation. The angles at B and C being adjacent to given things, are found by Equa. 1., and the hypotenuse BC by Equa. 2.

1. Cos. AG = sin. AB : R : : cot. OC = tan. AC : tan. B, like AC. Cos. OC = sin. AC : R : : cot. AG = tan. AB : tan. C, like AB. 2. R : sin. OC = cos. AC : : sin. AG = cos. AB : cos. BC acute, for AB, AC are like.

AC 67° 30′	R + tan. 20·3827757	sin. 9·9656153	cos. 9·5828397
AB 47° 38′	sin. 9·8685548	R + tan. 20·0399770	cos. 9·8285778
B 72° 59′ 2″	tan. 10·5142209	C tan. 10·0743617	BC cos. 9·4114175
		C = 49° 52′ 53″	BC = 75° 3′ 21″

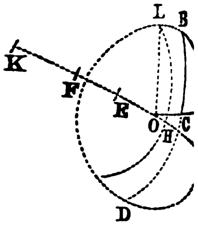
2. Given the sides about the right angle AB 108° 44′, and AC 67° 42′; to find the rest. Ans. The angles C 107° 25′ 13″, and B 68° 46′ 25″, and the hyp. BC 97°.

3. Given the sides about the right angle AC 127° 48′, and AB 71° 25′; to find the rest. Ans. The angles B 126° 19′ 29″, and C 75° 7′ 21″, and the side BC 101° 15′ 49″.

CASE VI. WHEN THE TWO OBLIQUE ANGLES ARE GIVEN.

1. In the spherical triangle ABC, right-angled at A, are given the angles at B 39° 48′, and at C 67° 12′; to find the hypotenuse BC, and the sides AB and AC.

Construction. Draw any diameter of the primitive EOG. Make OE the semi-tangent, and OF the tangent of 39° 48′, and from F with its secant describe the circle BCD. Add and subtract the angles, and make OK the semi-tangent of their sum, and OH that of their difference; then upon the diameter HK describe a circle, cutting the primitive in L. Join LO, and draw OA perpendicular to it; then ABC is the triangle.



The hypotenuse and the sides are measured as before.

Calculation. The hypotenuse being adjacent to the given angles i by Equa. 1., and the sides by Equa. 2.

1. Tan. B : cot. C : : R : cos. BC acute, for B and C are al
2. Sin. B : R : : cos. C : sin. AG = cos. AB, like C; and si R : : cos. B : sin. OC = cos. AC, like B.

C 67° 12′	R + cot.	19·6236227		R + cos.	19·5882692		sin. 9
B 39° 48′	tan.	9·9207329		sin.	9·8062544		R + cos. 19
BC 59° 41′ 59″	cos.	9·7028898	AB	cos.	9·7820348	AC	cos. 9
			AB = 52° 44′ 35″			AC =	

2. Given the angles B 112° 38′, and C 63° 40′; the sides. Ans. The hyp. BC 101° 54′ 34″, the sides A 25′ 44″, and AB 61° 16′ 30″.

3. Given the angles C 102° 28′, and B 118° 30′; the sides. Ans. The hyp. BC 83° 6′ 20″, the sides A 13′ 11″, and AC 119° 15′ 14″.

SOLUTION OF OBLIQUE-ANGLED SPHE TRIANGLES.

When the three sides or the three angles are not th parts, the solution may always be obtained by drawin pendicular from the extremity of a given side and op given angle, and then computing by Napier's rules of cular parts.

The following Table contains the proportions for t tion of the 12 cases of oblique-angled spherical t ABC represents any spherical triangle in which the pe cular AD either falls within the triangle or meets the l produced beyond C.

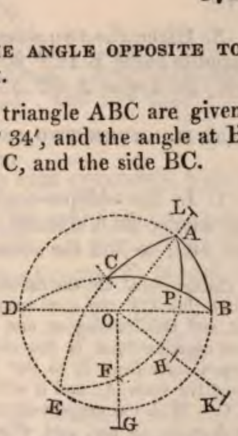
Note. The cases referred to are those of the p Table.

		AB.	cases, ACB is ambiguous.
2	AB, AC, and B, opposite to AC.	BC, the third side.	R : cos. B. : : tan. AB : tan. BD (case 2,) and cos. AB : cos. AC : : cos. BD : cos. DC. When ABC is acute, DC, CA are of the same affection, otherwise they are of different affection. If CD be not less than DB, their sum is CB; if CD be less than DB, but their sum not less than 180°, their difference is CB. In other cases, CB is ambiguous.
3	AB, AC, and B, opposite to AC.	A, the angle contained by the sides.	R : cos. AB : : tan B : cot. BAD (case 3,) and tan. AC : tan. AB : : cos. BAD : cos. DAC. If B be acute, DAC and AC are of the same affection, otherwise they are of different affection. If DAC be not less than BAD, their sum is BAC : if DAC be less than BAD, but their sum not less than 180°, their difference is BAC. In other cases, BAC is ambiguous.
4	B, C, and AB, two angles and the side opposite to one of them C.	AC, the side opposite to B.	Sin. C: sin. B : : sin. AB : sin. AC. If the sum of B and C be less than 180°, and B less than C, AC is acute : or if the sum of B and C be greater than 180°, and B greater than C, AC is obtuse. In other cases, AC is ambiguous.
5	B, C, and AB, two angles and the side opposite to one of them C.	A, the third angle.	R : cos. AB : : tan. B : cot. BAD, (case 3,) and cos. B : cos. C : : sin. BAD : sin. DAC, which is less than BAD, if B, C be of different affection, or less than the supplement of BAD, if B and C be of the same affection : In other cases it is ambiguous. When B and C are of the same affection, BAC is the sum of BAD, DAC, otherwise it is their difference.
6	B, C, and AB, two angles and the side opposite to one of them, C.	BC, the side between the angles.	R : cos. B : : tan. AB : tan. BD, (case 2,) and tan. C : tan. B : : sin. BD : sin. DC; and DC is less than DB, if B and C be of different affection; or less than the supplement of DB, if B and C be of the same affection. In other cases, DC is ambiguous. If B and C be of the same affection, BC is the sum of BD, DC; otherwise it is their difference.

Cases.	Given.	Sought.	Solution.
7	AB, BC, and B, two sides and the included angle.	C, one of the other angles.	R : cos. B : : tan. AB : tan. BD, (case 2,) and the difference of BC and BD is DC. And sin. DC : sin. DB : tan. B : tan. C, and B, C are of the same affection, if BC be greater than BD; otherwise they are of different affection.
8	AB, BC, and B, two sides and the included angle.	AC, the third side.	Find BD and DC as in the last case, then cos. BD : cos. DC : : cos. BA : cos. AC. If BD, DC be of the same affection, BA, AC are of the same affection; otherwise they are of different affection. Or add the sines of the two given sides, and twice the sine of half the contained angle, and from half the sum of these three logarithms subtract the sine of half the difference of the sides; the remainder is the tangent of an arc, whose sine taken from the half sum will leave the sine of half the required side.
9	A, B, and AB, two angles, and the included side.	C, the third angle.	R : cos. AB : : tan. B : cot. BAD, (case 3,) and the difference of BAC, BAD, is DAC, then sin. BAD : sin. DAC : : cos. B : cos. C, if BAC be greater than BAD, B, C are of the same affection; otherwise they are of different affection.
10	A, B, and AB, two angles and the included side.	AC, one of the other sides.	Find BAD and DAC, as in the last case; then cos. DAC : cos. BAD : : tan. AB : tan. AC. If DAC, and B be of the same affection, AC is less than 90°; otherwise it is greater than 90°.
11	AB, AC, and BC, the three sides.	B, one of the angles.	Let the perp. AD fall within, or be the nearest to B or C that falls without; then tan. $\frac{1}{2}$ BC : tan. $\frac{1}{2}$ sum of BA, AC : : tan. $\frac{1}{2}$ diff. of BA, AC : tan. $\frac{1}{2}$ E, and $\frac{1}{2}$ E added to $\frac{1}{2}$ BC, gives the segment nearest the greater side, if the sum of AB, AC be less than 180°; otherwise it gives the segment nearest the less side. And tan. AB : tan. BD : : R : cos. B (case 12.) Otherwise, Let D be $\frac{1}{2}$ the diff. of AB, BC; then the rect. sin. AB, sin. BC : rect. sin. sum and diff. of D and $\frac{1}{2}$ AC : : R^2 : : sin.^{2}, $\frac{1}{2}$ B. Otherwise, Let P be $\frac{1}{2}$ the perimeter; then rect. sin. AB, sin. BC : rect. sin. P, sin. diff. of P, AC : : R^2 : cos.^{2}, $\frac{1}{2}$ B.
12	A, B, and C, the	AC, one of	With the supplement of either of the angles A, C, and the measures of the other two

CASE I. GIVEN TWO SIDES, AND THE ANGLE OPPOSITE TO ONE OF THEM.

1. In the oblique-angled spherical triangle ABC are given the two sides AB 43° 30′, and AC 67° 34′, and the angle at B 72° 12′; to find the angles at A and C, and the side BC.



Construction. Draw he diameter of he primitive BOD, and OG perpendi-ular to it. Make OF the semi-tangent f 72° 12′, and OG its tangent, and rom G describe the circle BCD. Make AB 43° 30′, and draw OA. Lay the ecant of 67° 34′ on OA produced to L, nd with its tangent describe from L an rc, cutting BCD in C, and describe the ircle ACE; then ABC is the triangle.

Let K be the centre of ACE, join KO, then KO is the tangent of the angle BAC, or of its supplement. Lay the semi-tangent of it from O to H or the pole of ACE. A ruler laid from F to C will cut off an arc on he primitive between it and B equal to BC. Lines from C through F and H will cut off on the primitive the measure of the angle at C. Describe the circle AFE, which will be perpendicular to BC.

Calculation. The angle at C is found thus; Sin. CA : sin. AB : : sin. B : sin. C, which is acute, because AB is less than AC, and AB+AC less than 180°.

To find the other parts; first, find BP thus, R : cos. B : : tan. AB : tan. BP; then cos. AB : cos. AC : : cos. BP : cos. PC, which is acute, because AC is acute, the angle at B being acute. Then CB = BP + PC, because B and C are of the same affection.

Again, Sin. AC : sin. CB : : sin. B : sin. A.

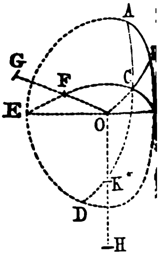
Sin. AB 43° 30′	9·8378122	Cos. B 72° 12′	9·4852888
Sin. B 72 12	9·9786960	Tan. AB 43 30	9·9772500
	19·8165082	Tan. BP 16° 10′ 38″	9·4625388
Sin. CA 67 34	9·9658243		
Sin. C 45° 9′ 31″	9·8506839		
Cos. AC 67° 34′	9·5816177	Sin. BC 75° 49′ 44″	9·9865787
Cos. BP 16 10 38″	9·9824542	Sin. B 72 12	9·9786960
	19·5640719		19·9652747
Cos. AB 43 30	9·8605622	Sin. AC 67 34	9·9658243
Cos. PC 59 39 6″	9·7035097	Sin. A 87 7 6″	9·9994504
BP 16 10 38		180	
BC 75 49 44		A 92 52 54 obtuse value.	

2. Given the sides AB 80° 5′, and AC 70° 10½′, and the angle B 33° 15′; to find the rest. Ans. The angles C 31° 34′ 32″, and A 161° 25′ 19″, and the side BC 145° 4′ 59″.

3. Given the two sides AB 114° 30′, AC 56° 40′, a[illegible] opposite angle B 125° 20′; to find the rest. Ans. BC [illegible] 49″, the angles A 62° 53′ 59″, and C 48° 30′ 25″.

CASE II. GIVEN TWO ANGLES, AND THE SIDE OPPOSITE ONE OF THEM.

1. In the oblique-angled spherical triangle ABC are [illegible] the angles at A 57° 36′, and at B 70° 34′, and the side [illegible] 85° 48′; to find the sides BC and BA, and the angle at [illegible]



Construction. On the radius of the primitive lay OF the semi-tangent of 57° 36′, and OG its tangent, and with its secant describe from G the circle ACD. Lay a ruler from E to 85° 48′ on the primitive from A, and it will cut ACD in C. With the tangent of 70° 34′ from O describe an arc, and with its secant from C cut that arc in H, from which as a centre describe the circle BCE; and ABC is the triangle.

On OH lay the semi-tangent of 70° 34′, to K. A ruler from K through C will cut off an arc of the primitive from B equal to BC. A ruler [illegible] C through K and F will mark off on the primitive the measure of [illegible] angle ACB. The radius OCL is the perpendicular on AB.

Calculation. The side CB is found thus; Sin. B : sin. A : : sin. [illegible] : sin. CB, acute, because A is less than B.

In the right-angled triangle ACL are given AC and the angle [illegible] to find AL and ACL. R : cos. A : : tan. AC : tan. AL acute. [illegible] R : cos. AC : : tan A : cot. ACL acute. Then tan. B : tan. [illegible] sin. AL : sin. BL; and cos. A : cos. B : : sin. ACL : sin. [illegible] Then AB = AL + LB, and ACB = ACL + LCB.

A	57° 36′	sin.	9·9265112	cos. A	9·7290244	tan.—R	0·1[illegible]
AC	85 48	sin.	9·9988321	tan. — R	1·1340945	cos.	8·8[illegible]
			19·9253433	AL tan.	10·8631189	ACL cot.	9·[illegible]
B	70 34	sin.	9·9745252	AL =	82° 11′ 46″	ACL =	83°[illegible]
CB	63 14 38	sin.	9·9508181				

B	70° 34′	cos.	9·5220656	A	57° 36′	tan.	10·1[illegible]
ACL	83 25 1″	sin.	9·9971270	AL	82 11 46″	sin.	9·9[illegible]
			19·5191926				20·1[illegible]
A	57 36	cos.	9·7290244	B	70 34	tan.	10·4[illegible]
BCL =	38 5 7″	sin.	9·7901682	BL =	33 25 16″	sin.	9·7[illegible]
ACL =	83 25 1			AL =	82 11 46		
ACB =	121° 30′ 8″			AB =	115 37 2		

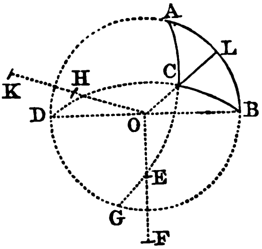
2. Given the angles A 115° 12′, and B 63° 30′, and side BC 122° 16′; to find the rest. Ans. The sides AB 1[illegible] 44′ 43″, AC 56° 45′ 16″, and the angle C 96° 18′ 59″.

3. Given the two angles B 91° 26′ 44″, C 102° 5′ 54″, and the side AC 118° 2′ 14″; to find the rest. Ans. The sides BC 23° 57′ 13″, and AB 120° 18′ 33″, and the angle A 27° 22′ 34″.

CASE III. GIVEN TWO SIDES AND THE INCLUDED ANGLE.

1. In the oblique-angled spherical triangle ABC are given the sides AB 58° 24′, and BC 67° 48′, and the included angle ABC 63° 43′; to find the angles at A and C, and the side AC.

Construction. On the radius of the primitive make OE the semi-tangent of 63° 43′, and OF its tangent, and with its secant from F describe the circle BCD. Make BA 58° 24′. A ruler laid from E to a point in the primitive, 67° 48′ from B, will cut BCD in C. Then describe the great circle ACG, and ACB is the triangle.



The distance OK of O from the centre of ACG is the tangent of the angle BAC, or its supplement. Make OH its semi-tangent. A line from H through C will cut off on the primitive from A the measure of AC, and lines from C through E and H will cut off on the primitive the measure of ACB.

The radius OCL is perpendicular to AB.

Calculation. In the triangle BCL, right-angled at L, are given the side CB 67° 48′, and the angle at B 63° 43′; to find BL and BCL. First R : cos. B : : tan. CB : tan. BL, and the difference of BL and BA is AL. In like manner, R : cos. BC : : tan. B. : cot. C. Then Sin. AL : sin. BL : : tan. B : tan. A, which is acute if BL be less than BA; otherwise it is obtuse. Also Cos. BL : cos. LA : : cos. BC : cos. CA, which is acute, or like BC, if BL, LA be of the same affection, otherwise obtuse. Also Tan. BL : tan. LA : : tan. BCL : tan. ACL, and ACB = BCL — ACL.

BC 67° 48′	tan.—R = 0·3892414	AL 11° 3′ 49″	cos. 9·9918525
B 63 43	cos. = 9·6462178	BC 67 48	cos. 9·5773088
BL 47 20 11″	tan. 10·0354592		19·5691613
BA 58 24		BL 47 20 11	cos. 9·8310329
AL 11 3 49		AC 56 49 35	cos. 9·7381284

BC 67° 48′	cos. 9 5773088	AL 11° 3′ 49″	tan. 9·2912194
B 63 43	tan.—R 0 3063883	BCL 52 34 54	tan. 10 1163027
BCL 52 34 54″	cot. 9·8856971		19·4075221
ACL 13 15 13		BL 47 20 11	tan. 10 0354584
ACB 65 50 7		ACL 13 15 13	tan. 9·3720637

BL 47° 20′ 11″	sin. =	9·8664912	
B 63 43	tan.	10·3063883	
		20·1728795	
AL 11 3 49	sin.	9·2830720	
BAC 82 39 29	tan.	10·8898075	

2. Given the sides AB 41° 9′ 46″, and BC 50° 5′ 47″, and the angle at B 114° 7′ 30″; to find the rest. Ans. AC 73° 56′ 40″, the angles at C 38° 41′ 21″, and at A 46° 45′ 49″.

3. Given the two sides AB 61° 14′, BC 58° 27′, and the included angle B 57° 53′ 55″; to find the rest. Ans. The angles C 77° 22′ 21″, and A 71° 33′ 30″, and the side AC 49° 33′.

CASE IV. GIVEN TWO ANGLES, AND THE INCLUDED SIDE.

1. In the oblique-angled spherical triangle ABC are given the side AB 75° 40′, and the angles at A 39° 38′, and B 58° 22′; to find the sides AC and BC, and the angle at C.

Construction. On the primitive make AB 75° 40′, and draw the diameters AD and BE, and perpendicular to them draw OG and OK. Lay the semi-tangent of 39° 38′ from O to F, and its tangent from O to G. Also lay the semi-tangent of 58° 22′ from O to H, and its tangent from O to K; and from the centres G and K describe the circles ACD and BCE; then ABC is the triangle.

The unknown parts are measured as before.

Describe the circle AHD, which is perpendicular to BC.

Calculation. In the triangle ABL, right-angled at L, are given the side AB, and the angle at B; to find the angle BAL. And the difference between BAL and BAC is CAL; thus, R : cos. BA : : tan. B : cot. BAL 68° 7′ 53″, whence CAL is 28° 29′ 53″. Then Sin. BAL : sin. CAL : : cos. B : cos. C, which is acute if BAC be greater than BAL; otherwise it is obtuse. Also cos. CAL : cos. BAL : : tan. AB : tan. AC, acute, if B and CAL be like. Lastly, Sin. B : sin. A : : sin. AC : sin. BC.

BA 75° 40′	cos. 9·3936852	CAL 28° 28′ 20″	sin.	9·6782749
B 58 22	tan.—R. 0·2104148	B 58 22	cos.	9·7197300
BAL 68 6 20″	cot. 9·6041000			19·3980049
BAC 39 38		BAL 68 6 20	sin.	9·9674883
CAL 28 28 20		74 22 1	cos.	9·4305166
		BCA 105 37 59		

BAL 68° 6′ 20″	cos. 9·5715898	A 39° 38′	sin. 9·8047336
AB 75 40	tan. 10·5925811	AC 58 56 16	sin. 9·9327818
	20·1641709		19·7375154
CAL 28 28 20	cos. 9·9440128	B 58 22	sin. 9·9301448
AC 58 56 16	tan. 10·2201581	BC 39 55 23	sin. 9·8073706

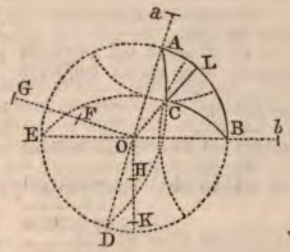
2. Given the side AB 124° 12′, and the angles at A 126° 20′, and at B 56° 15′; to find the rest. Ans. The sides AC 43° 30′ 31″, and BC 138° 9′ 42″, and the angle C 92° 42′ 46″.

3. Given the two angles A 58° 5′ 4″, B 62° 34′ 6″, and the side AB 122°; to find the rest. Ans. The angle C 130°, the sides AC 79° 17′ 14″, and BC 70°.

CASE V. WHEN THE THREE SIDES ARE GIVEN.

1. In the oblique-angled spherical triangle ABC are given the sides AB 82° 26′, BC 68° 53′, and AC 57° 30′; to find the angles.

Construction. On the primitive make AB 82° 26′, and draw the diameters AOD, BOE, and make O *a* and O *b* the secants of 57° 30′ and of 68° 53′, and with the tangents of these arcs from *a* and *b* describe circles cutting one another in C, and describe the circles BCE and ACD; then ABC is the triangle.



The distances from O of the centres of ACD and BCE, measured on the line of tangents, give the angles at A and B, and the angle at C is measured as before.

The radius OCL is the perpendicular upon AB.

Calculation. Tan. ½ AB : tan. ½ (BC + CA) : : tan. ½ (BC—CA): tan. ½ (BL$\mp$LA), and ½ BA$\mp$ ½ (BL$\mp$LA) = BL. Then, Tan. BC : tan. BL : : R : cos. B and tan. CA : tan. AL : : R : cos. A, and sin. CA : sin. BA : : sin. B : sin. C.

½ (BC+AC) 63° 11′ 30″	tan. 10.2964344	BL53° 54′ 22″ tan.+R.	20.1372432
½ (BC—AC) 5° 41′ 30″	tan. 8·9985482	BC68° 53′ tan.	10.4131853
	19·2949826	ABC58° 0′ 45″ cos.	9·7240579
½ BA 41° 13′	tan. 9·9424782		
½ (BL—AL) 12° 41′ 22″	tan. 9·3525044	AL28° 31′ 38″ tan.+R.	19.7352564
BL = 53° 54′ 22″		AC57° 30′ tan.	10.1958127
AL = 28° 31′ 38″		BAC69° 44′ 21″ cos.	9·5394437

AB 82° 26′	sin.	9·9962017
B 58° 0′ 45″	sin.	9·9284794
		19·9246814
AC 57° 30′	sin.	9·9260292
85° 29′ 18″	sin.	9·9986522
94° 30′ 42″ = the angle ACB.		

METHOD II. From ½ the sum of the three sides take the side opposite to the angle sought; and add the arithmetical complements of the sines of the two containing sides, and the sines of the ½ sum and remainder; and ½ the sum of these four is the cosine of ½ the angle sought.

METHOD III. Take the sum and difference of ½ the base, and ½ the difference of the sides, and then add the sines of this sum and difference, and the arithmetical complements of the sines of the containing sides; and ½ the sum of these four is the sine of ½ the angle sought.

NOTE. Instead of taking the sum and difference of ½ the base, and ½ the difference of the sides, the two containing sides may be subtracted from the ½ sum of the three sides.

METHOD IV. Add the arithmetical complements of the sines of the half sum, and of its excess above the base, and the sines of its excesses above the other two sides; and ½ the sum of these four is the tangent of ½ the angle sought.

NOTE. In using the common tables of logarithms, the third method is more accurate than the second when the required angle is small, and the second is more accurate when it is large. The fourth method may be used in all cases, except when the angle sought is very nearly equal to two right angles.

BY THE 2D METHOD.

AB =	82° 26′	
BC =	68 53	ar. co. sin. 0·0301883
AC =	57 30	ar. co. sin. 0·0739708
Sum	208 49	
½ Sum	104 24 30	sin. 9 9861207
	82 26	
Diff.	21 58 30	sin. 9·5731061
		½)19·6638864
	47 15 21·6	cos. 9·8316932
	2	
	94 30 43·2 Angle at C.	

BY THE 3D METHOD.

½ sum	104° 24′ 30″	
BC	68 53	ar. co. sin. 0·0301888
AC	57 30	ar. co. sin. 0·0739708
1st rem.	35 31 30	sin. 9·7642196
2d rem.	46 54 30	sin. 9 8634785
		½)19 7318577
	47 15 21·6	sin. 9·8659288
	2	
	94 30 43·2 Angle at C.	

BY THE 4TH METHOD.

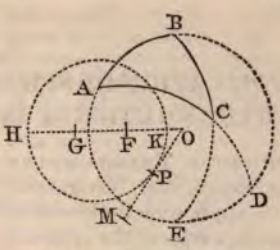
Half sum	104° 24′ 30″	ar. co. sin.	0·0138793
Exc. above AB	21 58 30	ar. co. sin.	0·4268939
Exc. above BC	35 31 30	sin.	9·7642196
Exc. above AC	46 54 30	sin.	9 8634785
		½	)20 0684713
	47 15 21.6	tan.	10·0342356
	2		
	94 30 43·2 Angle ACB.		

2. Given the sides AC 50° 54′ 32″, CB 37° 47′ 18″, and AB 74° 51′ 50″; to find the angles. Ans. The angles at B 44° 10′ 40″, at A 33° 22′ 45″, and at C 119° 55′ 6″.

3. Given AB 58° 0′ 5″, AC 88° 12′ 28″·8, and BC 94° 52′ 40″·8; to find the angles. Ans. The angles at C 57° 40′ 21″·6, at B 84° 49′ 2″, and at A 96° 33′ 28″.

CASE VI. WHEN THE THREE ANGLES ARE GIVEN.

1. *In the oblique-angled spherical triangle* ABC *are given the angles* A 78° 25′, B 110° 30′, and C 64° 48′; to find the sides.



Construction. Draw the radius of the primitive, and make OF the semi-tangent of 69° 30′=180°—110° 30′, and make OG its tangent, and with its secant describe from G the circle BCE. Lay the semi-tangent of 134° 18′=69° 30′+64° 48′ from O to H, and the semi-tangent of 4° 42′=69° 30′—64° 48′ from O the same way to K, and upon the diameter HK describe the circle HPK, and with the semi-tangent of 78° 25′ from O cut that circle in P. Join OP, and on it lay OM the tangent of 78° 25′, and with the secant of 78° 25′ from M describe the circle ACD; then ABC is the triangle.

Describe a great circle through the points F and P. The triangle OFP is semi-supplemental to ABC. For OP is the measure of BAC, because O and P are the poles of AB and AC; FP is the measure of ACB, because F and P are the poles of BC and CA, and OF is the supplement of ABC. Also, AB is the measure of the angle POF, because A and B are the poles of PO and OF; and BC is the measure of OFP, because B and C are the poles of OF and FP, and AC is the supplement of OPF.

Calculation. To find BC or the angle OFP. Take OF the supplement of ABC 69° 30′, and the difference between it and C or PF is 4° 42′, and the half of it taken from ½ BAC or OP, and added to it, are 36° 51′½ and 41° 33′½. Add the arithmetical complements of the sines of CBD and of C, and the sines of 41° 33′½ and of 36° 51′½; and ½ the sum of these four is the sine of ½ OFP or of ½ BC. In the same manner, we find AB and AC.

180—B 69° 30′ ar. co.	sin. 0·0284124
C 64 48 ar. co.	sin. 0·0434344
½ sum 106° 21′ 30″	sin. 9·9820536
A 78 25	
27 56 30	sin. 9·6707767
	2)19·7246771
43° 15′ 7″	cos. 9·8623385,
BC 86 30 14	

A 78° 25′ ar. co.	sin. 0·0089363
180–B 69 30 ar. co.	sin. 0·0284124
½ sum 106 21½	sin. 9·9820536
C 64 48	
41 33½	sin. 9·8217638
	2)19·8411661
33° 36′ 15″	cos. 9·9205830
BC 67 12 30	

180—A 101° 35′ arith. comp.	sin. 0·0089363
C 64 48 arith. comp.	sin. 0·0434344
½ sum 138 26 30	sin. 9·8217638
B 110 30	
27 56 30	sin. 9·6707767
	2)19·5449112
53 41 17	cos. 9·7724556
AC 107 22 34	

2. Given the angles A 44° 10′ 40″, B 33° 22′ 45″, and C 119° 55′ 6″; to find the sides. Ans. The sides BC 50° 54′ 30″·8, AB 74° 51′ 46″·3, and AC 37° 47′ 17″·5.

3. Given the three angles A 87° 46′ 13″, B 46° 34′ 5″ and C 53° 39′ 20″; to find the sides. Ans. The sides AB 31° 24′, BC 40° 16′, and AC 28° 1′.

APPLICATION OF SPHERICAL TRIGONOMETRY TO THE SOLUTION OF ASTRONOMICAL PROBLEMS.

Spherical Trigonometry is of great use in Astronomy, Geography, and Navigation; and therefore a few examples of its application to these sciences are given here, after explaining the circles of the sphere.

To lay down the circles of the sphere on the plane of the meridian of Edinburgh, in Lat. 55° 57′ 20″ N.

Let the primitive be the meridian. Draw the diameter HR for the horizon, and the perpendicular diameter ZN; then Z is the zenith, and N the nadir. Make RP, ZE, each 55° 57′ 20″, and draw the diameters P*p* and EQ; then P and *p* are the poles, and EQ the equator, and P*p* the hour-circle of six. About the points P and *p* as poles describe the circles *d e f* and *g h k* for the polar circles at the distance of 23° 28′; and in the same manner describe the tropics about the poles P and *p* at the distance of 66° 32′.

Suppose the time for which the circles are drawn to be the 3d August, 1831, at 9h 36m in the morning. The declination for that time is 17° 40′ N. About the pole P, at the distance of 72° 20′, describe the circle *a* C *b*, which is the parallel of the sun's declination for that day. Let it meet HR in A, P*p* in C, and ZN in F, and describe the great circles PA*p*, PF*p*, meeting EQ in B, G, and ZCN meeting HR in D. Describe the great circle PS*p*, making the angle ZPS 36° = 2h 24m the time from noon, and describe the circle ZSN meeting HR in T, and let PS*p* meet EQ in M.

The point *b* is the sun's place at midnight, and *a* his place at noon; A the point where he rises, C is his place at six, F his place when due east, and S his place at the given time. The circle ZON is the prime, or east and west vertical circle; O the east or west point of the horizon, R its north, and H its south points; R*b* is the sun's depression at midnight, *a*H is his meridian altitude, ST his altitude at the given time, OF his altitude when east, and CD his altitude at six. The arch QB, or the angle QPB, is the time of the sun's rising from midnight, and BO or BPO the time from six, which is called the sun's ascensional difference; BE, or BPE, the time of his rising from noon; OG, or OPG, the time from six, when he is due east; and GE, or

GPE, the time from noon. Also OM, or OPM, is the given time from six, and EM, or EPM, the given time from noon. AR, or AZR, is the sun's amplitude from the north; OA, or OZA, his amplitude from the east; and AH, or AZH, from the south. RD, or RZD, is his azimuth from the north at six; and DH, or DZH, from the south. And HT, or HZT, is his azimuth from the south at the given time; and TR, or TZR, that from the north.

PROBLEM I. Given the obliquity of the ecliptic, and the sun's longitude, to find his declination and right ascension.

In the spherical triangle SMO, right-angled at M, are given the angle SOM, the obliquity of the ecliptic, and the side SO the sun's longitude; to find SM the declination, and OM the right ascension. These are found by Case 1. Right-angled Triangles.

PROBLEM II. Given the latitude of the place, and the sun's declination, to find his amplitude, and the time of his rising.

In the spherical triangle APR, right-angled at R, are given PR the latitude, and the hypotenuse PA, the polar distance $= 90° \pm$ the sun's declination; to find AR the amplitude from the north, and the angle APR, which, converted into time at the rate of 15° to an hour, gives the time from midnight when the sun rises. Wherefore, by Case 4. of right-angled spherical triangles, cos. Latitude : R : : cos. polar dist. or sin. decl. : cos. RA, or sin. OA, the amplitude.

And tan. polar dist. : tan. Lat. : : R : cos. P, the time of rising.

The same things may be found in the triangle OAB right-angled at B, where AOB or RQ is the co-latitude, and AB the declination; to find AO the amplitude, and OB the ascensional difference, which, subtracted from six hours, gives the time of sun-rising. This is wrought by Case 3.

PROBLEM III. The same things being given, to find the sun's azimuth and altitude at six o'clock.

In the triangle PCZ, right-angled at P, are given PZ the co-latitude, and PC the polar distance; to find ZC the zenith distance, or complement of CD the altitude, and the angle CZP the azimuth from the north. Wherefore, by Case 5. R : cos. ZP = sin. Lat. : : cos. CP = sin. declination : cos. CZ, or sin. CD the altitude.

And sin. ZP = cos. Lat. : R : : tan. PC, or cot. decl. : tan. Z, the azimuth.

The same things may be found in the triangle OCD, right angled at D, where COD or PR is the latitude, and OC the declination. This is wrought by Case 1.

PROBLEM IV. The latitude and declination being still given, to find the sun's altitude, and the time when he is east.

In the triangle ZPF, right-angled at Z, are given ZP the co-latitude, and PF the polar distance; to find ZF the zenith distance, and the angle ZPF the time from noon. Wherefore, by Case 4. cos. ZP, or sin. Lat. : R : : cos. PF, or sin. decl. : cos FZ, or sin. FO, the altitude. And tan. FP, or cot. decl. : tan. ZP, or cot. Lat. : : R : cos. P.

The same things may be found in the triangle FOG, right-angled at G, in which are given FOG, or ZE, the latitude, and FG the declination; to find FO the altitude, and OG the complement of GE, the time from noon. This is wrought by Case 3.

Problem V. Given the latitude, declination, and hour; to find the sun's altitude and azimuth at that time.

In the triangle OSM, right-angled at M, are given MS the declination, and MO the time from six, to find the angle MOS (by Case 5.) sin. OM : R : : tan. MS : tan. O, and SOM + colat. EOH = SOT. Also R : cos. MO : : cos. MS : cos. SO. Then in the triangle OST, right-angled at T, are given SO, and the angle SOT; to find OT, the complement of TH, the azimuth, and TS the altitude, by Case 1.

The same things may be found by resolving the oblique-angled triangle PZS, in which are given PZ the co-latitude, PS the polar distance, and the angle ZPS the hour from noon; to find ZS the zenith distance, and the angle at Z the azimuth, by Case 3. of oblique-angled spherical triangles.

Problem VI. Given the latitude and longitude of the moon, or of a star, and the obliquity of the ecliptic; to find the right ascension and declination.

Suppose HR the equator, and EQ the ecliptic, then the latitude of the moon or any star S is MS, the longitude OM, the right ascension OT, the declination TS, and the obliquity of the ecliptic TOM. Therefore in the triangle OMS, right-angled at M, are given the two sides OM and MS about the right angle, to find the side OS and the angle MOS, which are found by Case 5. Now it is evident that when the moon or star S is without the ecliptic, the angle MOS added to the obliquity of the eliptic will give the angle TOS, or when S is within the ecliptic, the difference of these angles will be TOS. Hence in the triangle OTS, right angled at T, are given the angle TOS, and the hypotenuse OS, to find the sides OT and TS, which are done by Case 1.

Problem VII. Given the latitude of the place, and the sun's declination; to find the time when twilight begins and ends.

This problem is solved by Case 5. Oblique-angled Triangles, since there are given the polar distance, the co-latitude, and the zenith distance $= 90° + 18°$, which form the three sides of an oblique-angled spherical triangle, from whence to find the angle at the pole opposite the zenith distance, which is the time from noon that twilight begins and ends.

Problem VIII. Given the right ascensions and declinations, or the longitudes and latitudes of two celestial objects; to find their distance.

This problem is solved by Case 3. Oblique-angled Triangles, since there are given two sides and the contained angle to find the opposite side. The sides are the complements of the declinations, or latitudes, and the contained angle the difference between the right ascensions, or longitudes. By this problem the distances of two places on the globe may be found, of which the latitudes and longitudes are given; for the

polar distances are the sides of the spherical triangle, and the difference of longitude is the measure of the contained angle.

NOTE. Astronomical observations require to be corrected for the effects of Refraction, Parallax, &c.; but as these belong entirely to practical astronomy, it would be improper to introduce tables and rules for them here. The student who wishes to obtain complete information on these subjects is referred to the Introduction to Galbraith's Mathematical and Astronomical Tables,—a work replete with the most valuable scientific instruction.

PROMISCUOUS EXERCISES.

1. On the 1st of April, 1831, the obliquity of the ecliptic being 23° 27′ 34·1″, and the sun's declination 4° 21′ 51″ N., Required his longitude and right ascension.

Ans. Longitude 11° 1′ 10·9″; right ascension 0h 40m 30·8sec.

2. On the 1st of July, 1831, the obliquity of the ecliptic being 23° 27′ 33·7″, and the sun's right ascension 6h 38m 27·3sec., Required his longitude and declination.

Ans. Longitude 3^S 8° 49′ 55″; declination 23° 9′ 54″ N.

3. On the 1st of January, 1831, the obliquity of the ecliptic being 23° 27′ 33″, and the sun's longitude 9^S 10° 23′ 58″, Required his right ascension and declination.

Ans. Right ascension 18h 45m 15·2sec.; declination 23° 3′ 5″ S.

4. On the 1st of October, 1831, the declination of the sun being 2° 59′ 38″ S., and his right ascension 12h 27m 41·3sec., Required his longitude, and the obliquity of the ecliptic.

Ans. Longitude 6^S 7° 32′ 20″; obliquity of the ecliptic 23° 27′ 34·9″.

5. On the 31st of December, 1831, the sun's longitude being 9^S 9° 7′ 50″, and his declination 23° 8′ 42″ S., Required his right ascension, and the obliquity of the ecliptic.

Ans. Right ascension 18h 39m 45sec.; obliquity 23° 27′ 35″.

6. On the 1st of April, 1831, the sun's longitude being 11° 1′ 10·9″, and his right ascension 40m 30·8sec., Required his declination, and the obliquity of the ecliptic.

Ans. Declination 4° 21′ 51″ N.; obliquity 23° 27′ 34·1″.

7. On the 1st of February, 1831, the sun's declination being 17° 13′ 14″ S., Required the time of his rising and amplitude on the parallel of Edinburgh, (55° 57′ 20″ N.)

Ans. Amplitude 58° 4′ 28″; time of rising 7h 49m 13½sec.

8. On the 1st of April, 1831, the sun's declination being 4° 21′ 51″ N., In what latitude does he rise at 9 o'clock?

Ans. Latitude 83° 50′ 24″ S.

9. On the 1st of May, 1830, the sun rises at Paris, in latitude 48° 50′ 14″ N., at 4h 48m 35sec. Required his declination. Ans. Declination 15° 0′ 20″ N.

10. On the 22d of June, 1831, the sun's declination being 23° 27′ 33″ N., Required his altitude at Edinburgh at 6 o'clock. Ans. Altitude 19° 15′ 42·4″.

11. The same things being given as in the last exercise, Required his altitude at 10 o'clock morning. Ans. 50° 46′ 5″.

12. Given the altitude of the sun 45° 32′, declination as in the last. Required the hour of the day at Edinburgh.

Ans. 9h 13m 28sec. morning, or 2h 46m 32sec. afternoon.

13. Given the sun's declination 15° 30′ 20″ N. Required his azimuth at 9 o'clock morning for Edinburgh.

Ans. 58° 39′ 24″.

14. Given the altitude of the sun at 6 o'clock 18° 30′ 15″. Required his azimuth for Edinburgh. Ans. 76° 55′ 52·6″.

15. On the 1st of August, 1831, the sun's declination being 18° 10′ 22″ N., Required the hour when he is due east at Edinburgh.

Ans. 6h 51m 15sec. morning, or 5h 8m 45sec. afternoon.

16. On the 10th of September, 1831, the sun's declination being 5° 8′ 26″ N., and his altitude when due east 16° 53′ 10″, Required the latitude of the place. Ans. Latitude 17° 58′ N.

17. On the 20th of January, 1831, the moon's longitude at noon, on the meridian of Greenwich, being 19° 11′ 27″, her latitude 3° 52′ 31″ S., and the obliquity of the ecliptic 23° 27′ 33·4″, Required her right ascension and declination.

Ans. Right ascension 19° 10′ 47″; declination 3° 55′ 53″ N.

18. On the 24th of May, 1831, the right ascension of the moon, on the meridian of Greenwich, at noon, being 217° 59′ 6″, her declination 9° 55′ 4″ S., and the obliquity of the ecliptic 23° 27′ 34″, Required her latitude and longitude.

Ans. Latitude 4° 46′ 53″ N.; longitude 7^S 8° 49′ 17″.

19. On the 1st of July, 1831, the moon's latitude, on the meridian of Greenwich, at midnight, being 2° 55′ 31″ S., her right ascension 358° 20′ 53″, and the obliquity of the ecliptic 23° 27′ 33·7″, Required her declination and longitude.

Ans. Declination 3° 54′ 20″ S.; longitude 11^S 26° 55′ 45″.

20. On the 1st of January, 1831, the declination of Spica Virginis being 10° 16′ 32·9″ S., the right ascension 13h 16m 18sec., and the mean obliquity of the ecliptic 23° 27′ 42·1″, Required the longitude and latitude of the star.

Ans. Longitude 6^S 21° 29′ 0·8″, latitude 2° 2′ 27·5″ S.

21. On the 1st of January, 1831, the mean obliquity of the ecliptic being 23° 27′ 42·1″, the longitude of Aldebaran 2^{S} 7° 25′ 39·3″, and the latitude 5° 28′ 45·8″ S., Required his declination and right ascension.

Ans. Declination 16° 9′ 44·7″ N.; right ascension 4h 26m 14sec.

22. On the 1st of January, 1831, the mean obliquity of the ecliptic being 23° 27′ 42·1″, the declination of Pollux 28° 25′ 39″ N., and the latitude 6° 40′ $20\frac{1}{4}$″ N., Required his longitude and right ascension.

Ans. Longitude 3^{S} 20° 53′ 2″; right ascension 7h 34m 58sec.

23. On the 1st of April, 1830, at noon on the meridian of Greenwich, the longitude of the moon being 3^{S} 25° 44′ 54″, her latitude 4° 14′ 7″ S., and the longitude of the sun 11° 15′ 53″, Required the distance between them.

Ans. 104° 26′ 36″.

24. On the 28th of April, 1830, the distance between the sun and moon's centre being 74° 11′ 43″, the moon's longitude 3^{S} 21° 48′ 42″, and her latitude 4° 17′ 5″ S., Required the longitude of the sun. Ans. 1^{S} 7° 39′ 43″.

25. On the 27th of August, 1830, at noon on the meridian of Greenwich, the distance of the moon's centre from the sun's being 100° 10′ 41″, the moon's longitude 8^{S} 13° 52′ 41″, and the sun's longitude 5^{S} 3° 39′ 22″, Required the moon's latitude. Ans. 5° 16′ 34″ N.

26. On the 1st of January, 1830, at noon, on the meridian of Greenwich, the distance between the moon and Aldebaran being 64° 43′ 20″, the right ascension of the star 4h 26m 10·5sec., the declination 16° 9′ 37″ N., and the right ascension of the moon 2° 52′ 10″, Required the declination of the moon.

Ans. 13′ 3″ N.

27. On the 7th of January, 1830, at noon, on the meridian of Greenwich, the distance between the moon and Regulus being 61° 15′ 32″, the declination of the moon 18° 22′ 42″ N., her right ascension 86° 11′ 58″, and the declination of the star 12° 47′ 43″ N., Required his right ascension. Ans. 9h 59m 18·7sec.

28. On the 4th of January, 1831, at midnight on the meridian of Greenwich, the right ascension of the moon being 184° 10′ 39″, her declination 1° 6′ 27″ N., the right ascension of Antares 16h 19m, and his north polar distance 116° 2′ 44″, Required the distance between the moon and star.

Ans. 64° 21′ 23″.

29. On the 7th of January, 1831, at noon, on the meridian of Greenwich, the moon's latitude being 4° 30′ 58″ N., and her

longitude $7^s\ 3° 21' 9''$, the latitude of Jupiter 23′ S., and his longitude $9^s\ 26° 42'$, Required his distance from the moon.

Ans. 83° 24′.

30. On the 13th of June, 1831, at noon, on the meridian of Greenwich, the latitude of Jupiter being 0° 49′ S., his longitude $10^s\ 22° 21'$, the latitude of Saturn 1° 33′ N., and his longitude $4^s\ 26° 48'$, Required their distance.

Ans. 175° 29′ 28″.

31. At what time will twilight begin and end at Edinburgh on the 20th August, 1831, the sun's declination being 12° 38′ 9″ N.

Ans. 1h 44m 41sec. morning, and 10h 15m 19sec afternoon.

32. In what latitude, on the 1st of September, 1831, does the twilight begin at 3h 20m in the morning, the sun's declination being 8° 28′ 54″ N. Ans. 48° 38′ 56″ N.

33. At Edinburgh the twilight begins at 4h. in the morning. Required the declination of the sun.

Ans. 2° 1′ 28·5″ S.

34. The latitude of Edinburgh is 55° 57′ 20″ N., and the longitude 3° 10′ 21″ W., the latitude of the Cape of Good Hope is 34° 29′ S., and its longitude 18° 23′ 15″ E. Required the distance between them. Ans. 5537·229 geog. miles.

35. The latitude of Greenwich Observatory is 51° 28′ 38″ N., the latitude of Bombay Church is 18° 57′ 44″ N., and the longitude 72° 54′ 43″ E. Required the distance between them. Ans. 3882·2157 geog. miles.

36. The latitude of Batavia is 6° 9′ S., and its longitude 106° 51′ 45″ E., the latitude of the Royal Observatory of Paris is 48° 50′ 14″ N., and its longitude 2° 20′ 15″ E. Required the distance between them. Ans. 6250·014 geog. miles.

A

TABLE,

CONTAINING

THE LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

Numbers from 1 to 100 and their Logarithms, with their Indices.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

NOTE. In the following part of the Table the Indices are omitted, as they can be very easily supplied by the directions given in the Section on Logarithms, page 93.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	[illegible]
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	[illegible]
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	[illegible]
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	[illegible]
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	[illegible]
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	[illegible]
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	[illegible]
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	[illegible]
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	[illegible]
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	[illegible]
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	[illegible]
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	[illegible]
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	[illegible]
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	[illegible]
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	[illegible]
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	[illegible]
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	[illegible]
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	[illegible]
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	[illegible]
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	[illegible]
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	[illegible]
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	[illegible]
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	[illegible]
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	[illegible]
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	[illegible]
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	[illegible]
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	[illegible]
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	[illegible]
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	[illegible]
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	[illegible]
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	[illegible]
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	[illegible]
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	[illegible]
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	[illegible]
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	[illegible]
5	130334	130655	130977	131298	131619	131939	132260	132580	132900	[illegible]
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	[illegible]
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	[illegible]
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	[illegible]
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	[illegible]
140	146128	146438	146748	147058	147367	147676	147985	148294	148603	[illegible]
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	[illegible]
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	[illegible]
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	[illegible]
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	[illegible]
5	161368	161667	161967	162266	162564	162863	3161	3460	3758	[illegible]
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	[illegible]
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	[illegible]
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	[illegible]
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	[illegible]
150	176091	176381	176670	176959	177248	177536	177825	178113	178401	[illegible]
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	[illegible]
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	[illegible]
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	[illegible]
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	[illegible]
5	190332	190612	190892	191171	191451	191730	192010	192289	192567	[illegible]
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	[illegible]
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	[illegible]
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	[illegible]
9	201397	201670	201943	202216	202488	2761	3033	3305	3577	[illegible]
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
170	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806	228
1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
5	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	310056	310268	310481	310693	310906	311118	311330	311542	212
5	311754	311966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	320146	320354	320562	320769	320977	321184	321391	321598	321805	322012	207
210	322219	322426	322633	322839	323046	323252	323458	323665	323871	324077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	330008	330211	203
4	330414	330617	330819	331022	331225	331427	331630	331832	2034	2236	202
5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	340047	340246	199
9	340444	340642	340841	341039	341237	341435	341632	341830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9
220	342423	342620	342817	343014	343212	343409	343606	343802	343999	[illegible]
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	[illegible]
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	[illegible]
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	[illegible]
4	350248	350442	350636	350829	351023	351216	351410	351603	351796	[illegible]
5	2183	2375	2568	2761	2954	3147	3339	3532	3724	[illegible]
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	[illegible]
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	[illegible]
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	[illegible]
9	9835	360025	360215	360404	360593	360783	360972	361161	361350	[illegible]
230	361728	361917	362105	362294	362482	362671	362859	363048	363236	[illegible]
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	[illegible]
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	[illegible]
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	[illegible]
4	9216	9401	9587	9772	9958	370143	370328	370513	370698	[illegible]
5	371068	371253	371437	371622	371806	1991	2175	2360	2544	[illegible]
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	[illegible]
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	[illegible]
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	[illegible]
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	[illegible]
240	380211	380392	380573	380754	380934	381115	381296	381476	381656	[illegible]
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	[illegible]
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	[illegible]
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	[illegible]
4	7390	7568	7746	7923	8101	8279	8456	8634	8811	[illegible]
5	9166	9343	9520	9698	9875	390051	390228	390405	390582	[illegible]
6	390935	391112	391288	391464	391641	1817	1993	2169	2345	[illegible]
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	[illegible]
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	[illegible]
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	[illegible]
250	397940	398114	398287	398461	398634	398808	398981	399154	399328	[illegible]
1	9674	9847	400020	400192	400365	400538	400711	400883	401056	[illegible]
2	401401	401573	1745	1917	2089	2261	2433	2605	2777	[illegible]
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	[illegible]
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	[illegible]
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	[illegible]
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	[illegible]
7	9933	410102	410271	410440	410609	410777	410946	411114	411283	[illegible]
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	[illegible]
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	[illegible]
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3	9956	420121	420286	420451	420616	420781	420945	421110	421275	[illegible]
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	[illegible]
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	[illegible]
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	[illegible]
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	[illegible]
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	[illegible]
9	9752	9914	430075	430236	430398	430559	430720	430881	431042	[illegible]
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1	2969	3130	3290	3450	3610	3770	3930	4090	4249	[illegible]
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	[illegible]
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	[illegible]
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	[illegible]
5	9333	9491	9648	9806	9964	440122	440279	440437	440594	[illegible]
6	440909	441066	441224	441381	441538	1695	1852	2009	2166	[illegible]
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	[illegible]
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	[illegible]
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1	8706	8861	9015	9170	9324	9478	9633	9787	9941	450095	154
2	450249	450403	450557	450711	450865	451018	451172	451326	451479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
5	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
8	9392	9543	9694	9845	9995	460146	460296	460447	460597	460748	151
9	460898	461048	461198	461348	461499	1649	1799	1948	2098	2248	150
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1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
5	9822	9969	470116	470263	470410	470557	470704	470851	470998	471145	147
6	471292	471438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	477121	477266	477411	477555	477700	477844	477989	478133	478278	478422	145
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
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3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
9	9958	490099	490239	490380	490520	490661	490801	490941	491081	491222	140
310	491362	491502	491642	491782	491922	492062	492201	492341	492481	492621	140
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
5	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
6	9687	9824	9962	500099	500236	500374	500511	500648	500785	500922	137
7	501059	501196	501333	1470	1607	1744	1880	2017	2154	2291	137
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
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2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
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4	510545	510679	510813	510947	511081	511215	1349	1482	1616	1750	134
5	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
8	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
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1	9828	9959	520090	520221	520353	520484	520615	520745	520876	521007	131
2	521138	521269	1400	1530	1661	1792	1922	2053	2183	2314	131
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
5	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	530072	128
9	530200	530328	530456	530584	530712	530840	530968	531096	531223	1351	128
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2	4026	4153	4280	4407	4534	4661	4787	4914	5041	[illegible]
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	[illegible]
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6	9076	9202	9327	9452	9578	9703	9829	9954	540079	[illegible]
7	540329	540455	540580	540705	540830	540955	541080	541205	1330	[illegible]
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	[illegible]
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	[illegible]
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4	9003	9126	9249	9371	9494	9616	9739	9861	9984	[illegible]
5	550228	550351	550473	550595	550717	550840	550962	551084	551206	[illegible]
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	[illegible]
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	[illegible]
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	[illegible]
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	[illegible]
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3	9907	560026	560146	560265	560385	560504	560624	560743	560863	[illegible]
4	561101	1221	1340	1459	1578	1698	1817	1936	2055	[illegible]
5	2293	2412	2531	2650	2769	2887	3006	3125	3244	[illegible]
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	[illegible]
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	[illegible]
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	[illegible]
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	[illegible]
370	568202	568319	568436	568554	568671	568788	568905	569023	569140	[illegible]
1	9374	9491	9608	9725	9842	9959	570076	570193	570309	[illegible]
2	570543	570660	570776	570893	571010	571126	1243	1359	1476	[illegible]
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	[illegible]
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	[illegible]
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6	5188	5303	5419	5534	5650	5765	5880	5996	6111	[illegible]
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	[illegible]
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	[illegible]
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5	5461	5574	5686	5799	5912	6024	6137	6250	6362	[illegible]
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7	7711	7823	7935	8047	8160	8272	8384	8496	8608	[illegible]
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	[illegible]
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5	6597	6707	6817	6927	7037	7146	7256	7366	7476	[illegible]
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	[illegible]
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	[illegible]
8	9883	9992	600101	600210	600319	600428	600537	600646	600755	[illegible]
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4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
7	9594	9701	9808	9914	610021	610128	610234	610341	610447	610554	107
8	610660	610767	610873	610979	1086	1192	1298	1405	1511	1617	106
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
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1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
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4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
5	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
7	620136	620240	620344	620448	620552	620656	620760	620864	620968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	623249	623353	623456	623559	623663	623766	623869	623973	624076	624179	103
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
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3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	630021	630123	630224	630326	102
7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342	102
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
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4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
5	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	100
6	9486	9586	9686	9785	9885	9984	640084	640183	640283	640382	99
7	640481	640581	640680	640779	640879	640978	1077	1177	1276	1375	99
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
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1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
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3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
6	9335	9432	9530	9627	9724	9821	9919	650016	650113	650210	97
7	650308	650405	650502	650599	650696	650793	650890	0987	1084	1181	97
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
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4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
7	9916	660011	660106	660201	660296	660391	660486	660581	660676	660771	95
8	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
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2	4642	4736	4830	4924	5018	5112	5206	5299	5393	[illegible]
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	[illegible]
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	[illegible]
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	[illegible]
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4	5778	5870	5962	6053	6145	6236	6328	6419	6511	[illegible]
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	[illegible]
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	[illegible]
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9	680336	680426	680517	680607	680698	680789	680879	0970	1060	[illegible]
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6	6636	6726	6815	6904	6994	7083	7172	7261	7351	[illegible]
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	[illegible]
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	[illegible]
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8	4330	4414	4497	4581	4665	4749	4833	4916	50	[illegible]
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2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	720077	83
5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
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1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
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6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
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4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
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1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
5	9668	9743	9819	9894	9970	760045	760121	760196	760272	760347	75
6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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3	5669	5743	5818	5892	5966	6041	6115	6190	6264	[illegible]	[illegible]
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	[illegible]	[illegible]
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	[illegible]	[illegible]
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	[illegible]	[illegible]
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4	3786	3860	3933	4006	4079	4152	4225	4298	4371	[illegible]	[illegible]
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6	5246	5319	5392	5465	5538	5610	5683	5756	5829	[illegible]	[illegible]
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	[illegible]	[illegible]
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4	1037	1109	1181	1253	1324	1396	1468	1540	1612	[illegible]	[illegible]
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	[illegible]	[illegible]
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9	4617	4689	4760	4831	4902	4974	5045	5116	5187	[illegible]	[illegible]
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7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
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9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	799409	799478	799547	799616	799685	799754	799823	799892	799961	69
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2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	68
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
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2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	6
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	6
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	6
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7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	6
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	6
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	6
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4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	6
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	6
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	6
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	6
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	6
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	6
670	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	6
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	6
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	6
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	6
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	6
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	6
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4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	6
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	6
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	6
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	6
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	6
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	6
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2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	6
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5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	6
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8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	6
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4	7573	7634	7696	7758	7819	7881	7943	8004	8066
5	8189	8251	8312	8374	8435	8497	8559	8620	8682
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3	3090	3150	3211	3272	3333	3394	3455	3516	3577
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5	4306	4367	4428	4488	4549	4610	4670	4731	4792
6	4913	4974	5034	5095	5156	5216	5277	5337	5398
7	5519	5580	5640	5701	5761	5822	5882	5943	6003
8	6124	6185	6245	6306	6366	6427	6487	6548	6608
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5	860338	860398	860458	860518	860578	0637	0697	0757	0817
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7	1534	1594	1654	1714	1773	1833	1893	1952	2012
8	2131	2191	2251	2310	2370	2430	2489	2549	2608
9	2728	2787	2847	2906	2966	3025	3085	3144	3204
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4	5696	5755	5814	5874	5933	5992	6051	6110	61
5	6287	6346	6405	6465	6524	6583	6642	6701	676
6	6878	6937	6996	7055	7114	7173	7232	7291	73
7	7467	7526	7585	7644	7703	7762	7821	7880	79
8	8056	8115	8174	8233	8292	8350	8409	8468	85
9	8644	8703	8762	8821	8879	8938	8997	9056	91
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3	0989	1047	1106	1164	1223	1281	1339	1398	14
4	1573	1631	1690	1748	1806	1865	1923	1981	20
5	2156	2215	2273	2331	2389	2448	2506	2564	26
6	2739	2797	2855	2913	2972	3030	3088	3146	3
7	3321	3379	3437	3495	3553	3611	3669	3727	3
8	3902	3960	4018	4076	4134	4192	4250	4308	4
9	4482	4540	4598	4656	4714	4772	4830	4888	4
750	875061	875119	875177	875235	875293	875351	875409	875466	875
1	5640	5698	5756	5813	5871	5929	5987	6045	6
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3	6795	6853	6910	6968	7026	7083	7141	7199	7
4	7371	7429	7487	7544	7602	7659	7717	7774	7
5	7947	8004	8062	8119	8177	8234	8292	8349	8
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7	9096	9153	9211	9268	9325	9383	9440	9497	9
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8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
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6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
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9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	892095	892150	892206	892262	892317	892373	892429	892484	892540	892595	56
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3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
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8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
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1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
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5	900367	900422	900476	900531	0586	0640	0695	0749	0804	0859	55
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7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
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2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
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4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
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9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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2	9556	9610	9663	9716	9770	9823	9877	9930	9984	910037	53
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5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
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3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	[illegible]
840	924279	924331	924383	924434	924486	924538	924589	924641	924693	[illegible]
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	[illegible]
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7	7883	7935	7986	8037	8088	8140	8191	8242	8293	[illegible]
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	[illegible]
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	[illegible]
850	929419	929470	929521	929572	929623	929674	929725	929776	929827	[illegible]
1	9930	9981	930032	930083	930134	930185	930236	930287	930338	[illegible]
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5	1966	2017	2068	2118	2169	2220	2271	2322	2372	[illegible]
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	[illegible]
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	[illegible]
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	[illegible]
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	[illegible]
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4	6514	6564	6614	6665	6715	6765	6815	6865	6916	[illegible]
5	7016	7066	7117	7167	7217	7267	7317	7367	7418	[illegible]
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	[illegible]
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	[illegible]
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	[illegible]
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	[illegible]
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7	3000	3049	3099	3148	3198	3247	3297	3346	3396	[illegible]
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	[illegible]
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	[illegible]
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5567	5616	5665	5715	5764	5813	5862	5912	49
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7041	7090	7140	7189	7238	7287	7336	7385	49
7532	7581	7630	7679	7728	7777	7826	7875	49
8022	8070	8119	8168	8217	8266	8315	8364	49
8511	8560	8609	8657	8706	8755	8804	8853	49
8999	9048	9097	9146	9195	9244	9292	9341	49
949488	949536	949585	949634	949683	949731	949780	949829	49
9975	950024	950073	950121	950170	950219	950267	950316	49
950462	0511	0560	0608	0657	0706	0754	0803	49
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1435	1483	1532	1580	1629	1677	1726	1775	49
1920	1969	2017	2066	2114	2163	2211	2260	48
2405	2453	2502	2550	2599	2647	2696	2744	48
2889	2938	2986	3034	3083	3131	3180	3228	48
3373	3421	3470	3518	3566	3615	3663	3711	48
3856	3905	3953	4001	4049	4098	4146	4194	48
954339	954387	954435	954484	954532	954580	954628	954677	48
4821	4869	4918	4966	5014	5062	5110	5158	48
5303	5351	5399	5447	5495	5543	5592	5640	48
5784	5832	5880	5928	5976	6024	6072	6120	48
6265	6313	6361	6409	6457	6505	6553	6601	48
6745	6793	6840	6888	6936	6984	7032	7080	48
7224	7272	7320	7368	7416	7464	7512	7559	48
7703	7751	7799	7847	7894	7942	7990	8038	48
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59137	959185	959232	959280	959328	959375	959423	959471	48
9614	9661	9709	9757	9804	9852	9900	9947	48
30090	960138	960185	960233	960281	960328	960376	960423	48
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1041	1089	1136	1184	1231	1279	1326	1374	48
1516	1563	1611	1658	1706	1753	1801	1848	47
1990	2038	2085	2132	2180	2227	2275	2322	47
2464	2511	2559	2606	2653	2701	2748	2795	47
937	2985	3032	3079	3126	3174	3221	3268	47
410	3457	3504	3552	3599	3646	3693	3741	47
882	963929	963977	964024	964071	964118	964165	964212	47
354	4401	4448	4495	4542	4590	4637	4684	47
325	4872	4919	4966	5013	5061	5108	5155	47
96	5343	5390	5437	5484	5531	5578	5625	47
66	5813	5860	5907	5954	6001	6048	6095	47
36	6283	6329	6376	6423	6470	6517	6564	47
05	6752	6799	6845	6892	6939	6986	7033	47
73	7220	7267	7314	7361	7408	7454	7501	47
2	7688	7735	7782	7829	7875	7922	7969	47
9	8156	8203	8249	8296	8343	8390	8436	47
6	968623	968670	968716	968763	968810	968856	968903	47
3	9090	9136	9183	9229	9276	9323	9369	47
	9556	9602	9649	9695	9742	9789	9835	47
	970021	970068	970114	970161	970207	970254	970300	47
	0486	0533	0579	0626	0672	0719	0765	46
	0951	0997	1044	1090	1137	1183	1229	46
	1415	1461	1508	1554	1601	1647	1693	46
	1879	1925	1971	2018	2064	2110	2157	46
	2342	2388	2434	2481	2527	2573	2619	46
	2804	2851	2897	2943	2989	3035	3082	46
	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8
940	973128	973174	973220	973266	973313	973359	973405	973451	[illegible]
1	3590	3636	3682	3728	3774	3820	3866	3913	[illegible]
2	4051	4097	4143	4189	4235	4281	4327	4374	[illegible]
3	4512	4558	4604	4650	4696	4742	4788	4834	[illegible]
4	4972	5018	5064	5110	5156	5202	5248	5294	[illegible]
5	5432	5478	5524	5570	5616	5662	5707	5753	[illegible]
6	5891	5937	5983	6029	6075	6121	6167	6212	[illegible]
7	6350	6396	6442	6488	6533	6579	6625	6671	6717
8	6808	6854	6900	6946	6992	7037	7083	7129	7175
9	7266	7312	7358	7403	7449	7495	7541	7586	7632
950	977724	977769	977815	977861	977906	977952	977998	978043	[illegible]
1	8181	8226	8272	8317	8363	8409	8454	8500	8546
2	8637	8683	8728	8774	8819	8865	8911	8956	9002
3	9093	9138	9184	9230	9275	9321	9366	9412	9457
4	9548	9594	9639	9685	9730	9776	9821	9867	9912
5	980003	980049	980094	980140	980185	[illegible]	[illegible]	[illegible]	[illegible]
6	0458	0503	0549	0594	0640	0685	0730	0776	0821
7	0912	0957	1003	1048	1093	1139	1184	1229	1275
8	1366	1411	1456	1501	1547	1592	1637	1683	1728
9	1819	1864	1909	1954	2000	2045	2090	2135	2181
960	982271	982316	982362	982407	982452	982497	982543	982588	[illegible]
1	2723	2769	2814	2859	2904	2949	2994	3040	3085
2	3175	3220	3265	3310	3356	3401	3446	3491	3536
3	3626	3671	3716	3762	3807	3852	3897	3942	3987
4	4077	4122	4167	4212	4257	4302	4347	4392	4437
5	4527	4572	4617	4662	4707	4752	4797	4842	4887
6	4977	5022	5067	5112	5157	5202	5247	5292	5337
7	5426	5471	5516	5561	5606	5651	5696	5741	5786
8	5875	5920	5965	6010	6055	6100	6144	6189	6234
9	6324	6369	6413	6458	6503	6548	6593	6637	6682
970	986772	986817	986861	986906	986951	986996	987040	987085	987130
1	7219	7264	7309	7353	7398	7443	7488	7532	7577
2	7666	7711	7756	7800	7845	7890	7934	7979	8024
3	8113	8157	8202	8247	8291	8336	8381	8425	8470
4	8559	8604	8648	8693	8737	8782	8826	8871	8916
5	9005	9049	9094	9138	9183	9227	9272	9316	9361
6	9450	9494	9539	9583	9628	9672	9717	9761	9806
7	9895	9939	9983	990028	990072	990117	990161	990206	990250
8	990339	990383	990428	0472	0516	0561	0605	0650	0694
9	0783	0827	0871	0916	0960	1004	1049	1093	1137
980	991226	991270	991315	991359	991403	991448	991492	991536	991580
1	1669	1713	1758	1802	1846	1890	1935	1979	2023
2	2111	2156	2200	2244	2288	2333	2377	2421	2465
3	2554	2598	2642	2686	2730	2774	2819	2863	2907
4	2995	3039	3083	3127	3172	3216	3260	3304	3348
5	3436	3480	3524	3568	3613	3657	3701	3745	3789
6	3877	3921	3965	4009	4053	4097	4141	4185	4229
7	4317	4361	4405	4449	4493	4537	4581	4625	4669
8	4757	4801	4845	4889	4933	4977	5021	5065	5108
9	5196	5240	5284	5328	5372	5416	5460	5504	5547
990	995635	995679	995723	995767	995811	995854	995898	995942	995986
1	6074	6117	6161	6205	6249	6293	6337	6380	6424
2	6512	6555	6599	6643	6687	6731	6774	6818	6862
3	6949	6993	7037	7080	7124	7168	7212	7255	7299
4	7386	7430	7474	7517	7561	7605	7648	7692	7736
5	7823	7867	7910	7954	7998	8041	8085	8129	8172
6	8259	8303	8347	8390	8434	8477	8521	8564	8608
7	8695	8739	8782	8826	8869	8913	8956	9000	9043
8	9131	9174	9218	9261	9305	9348	9392	9435	9479
9	9565	9609	9652	9696	9739	9783	9826	9870	9913
N.	0	1	2	3	4	5	6	7	8

TABLES

OF

LOGARITHMIC SINES AND TANGENTS

FOR

EVERY DEGREE AND MINUTE

OF THE

QUADRANT;

AND OF

NATURAL SINES AND TANGENTS

FOR

EVERY FIVE MINUTES

OF A

DEGREE.

A TABLE OF THE ANGLES WHICH EVERY POINT AND Q[…]
POINT OF THE COMPASS MAKES WITH THE MERIDIA[…]

North.		Points.	° ′ ″	Points.	South.	
		0 ¼	2 48 45	0 ¼		
		0 ½	5 37 30	0 ½		
		0 ¾	8 26 15	0 ¾		
N. b. E.	N. b. W.	1	11 15 0	1	S. b. E.	S.b.
		1 ¼	14 3 45	1 ¼		
		1 ½	16 52 30	1 ½		
		1 ¾	19 41 15	1 ¾		
N.N.E.	N.N.W.	2	22 30 0	2	S.S.E.	S.S.
		2 ¼	25 18 45	2 ¼		
		2 ½	28 7 30	2 ½		
		2 ¾	30 56 15	2 ¾		
N.E. b. N.	N.W. b. N.	3	33 45 0	3	S.E. b. S.	S.W
		3 ¼	36 33 45	3 ¼		
		3 ½	39 22 30	3 ½		
		3 ¾	42 11 15	3 ¾		
N.E.	N.W.	4	45 0 0	4	S.E.	S
		4 ¼	47 48 45	4 ¼		
		4 ½	50 37 30	4 ½		
		4 ¾	53 26 15	4 ¾		
N.E. b. E.	N.W.b. W.	5	56 15 0	5	S.E. b. E.	S.W
		5 ¼	59 3 45	5 ¼		
		5 ½	61 52 30	5 ½		
		5 ¾	64 41 15	5 ¾		
E.N.E.	W.N.W.	6	67 30 0	6	E.S.E.	W.
		6 ¼	70 18 45	6 ¼		
		6 ½	73 7 30	6 ½		
		6 ¾	75 56 15	6 ¾		
E. b. N.	W. b. N.	7	78 45 0	7	E. b. S.	W.
		7 ¼	81 33 45	7 ¼		
		7 ½	84 22 30	7 ½		
		7 ¾	87 11 15	7 ¾		
East.	West.	8	90 0 0	8	East.	W

A TABLE OF LOGARITHMIC SINES, TANGENTS, AND SECA[…]
EVERY POINT AND QUARTER POINT OF THE COMPA[…]

Points.	Sine.	Cosine.	Tang.	Cotang.	Secant.	Cosec.
0	0.000000	10.000000	0.000000	Infinite.	10.000000	Infinite.
0 ¼	8.690796	9.999477	8.691319	11.308681	10.000523	11.30920[…]
0 ½	8.991302	9.997904	8.993398	11.006602	10.002096	11.00869[…]
0 ¾	9.166520	9.995274	9.171247	10.828753	10.004726	10.83348[…]
1	9.290236	9.991574	9.298662	10.701338	10.008426	10.70976[…]
1 ¼	9.385571	9.986786	9.398785	10.601215	10.013214	10.61442[…]
1 ½	9.462824	9.980885	9.481939	10.518061	10.019115	10.53717[…]
1 ¾	9.527488	9.973841	9.553647	10.446353	10.026159	10.47251[…]
2	9.582840	9.965615	9.617224	10.382776	10.034385	10.41716[…]
2 ¼	9.630992	9.956163	9.674829	10.325171	10.043837	10.36900[…]
2 ½	9.673387	9.945430	9.727957	10.272043	10.054570	10.32661[…]
2 ¾	9.711050	9.933350	9.777700	10.222300	10.066650	10.28889[…]
3	9.744739	9.919846	9.824893	10.175107	10.080154	10.2552[…]
3 ¼	9.775027	9.904828	9.870199	10.129801	10.095172	10.2249[…]
3 ½	9.802359	9.888185	9.914173	10.085827	10.111815	10.1976[…]
3 ¾	9.827084	9.869790	9.957295	10.042705	10.130210	10.1729[…]
4	9.849485	9.849485	10.000000	10.000000	10.150515	10.1505[…]
	Cosine.	Sine.	Cotang.	Tang.	Cosec.	Secan[…]

0 Degree.			1 Degree.				
Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
10.000000	0.000000	Infinite.	8.241855	9.999934	8.241921	11.758079	60
000000	6.463726	13.536274	249033	999932	249102	750898	59
000000	764756	235244	256094	999929	256165	743835	58
000000	940847	059153	263042	999927	263115	736885	57
000000	7.065786	12.934214	269881	999925	269956	730044	56
000000	162696	837304	276614	999922	276691	723309	55
9.999999	241878	758122	283243	999920	283323	716677	54
999999	308825	691175	289773	999918	289856	710144	53
999999	366817	633183	296207	999915	296292	703708	52
999999	417970	582030	302546	999913	302634	697366	51
999998	463727	536273	308794	999910	308884	691116	50
9.999998	7.505120	12.494880	8.314954	9.999907	8.315046	11.684954	49
999997	542909	457091	321027	999905	321122	678878	48
999997	577672	422328	327016	999902	327114	672886	47
999996	609857	390143	332924	999899	333025	666975	46
999996	639820	360180	338753	999897	338856	661144	45
999995	667849	332151	344504	999894	344610	655390	44
999995	694179	305821	350181	999891	350289	649711	43
999994	719003	280997	355783	999888	355895	644105	42
999993	742484	257516	361315	999885	361430	638570	41
999993	764761	235239	366777	999882	366895	633105	40
9.999992	7.785951	12.214049	8.372171	9.999879	8.372292	11.627708	39
999991	806155	193845	377499	999876	377622	622378	38
999990	825460	174540	382762	999873	382889	617111	37
999989	843944	156056	387962	999870	388092	611908	36
999988	861674	138326	393101	999867	393234	606766	35
999988	878708	121292	398179	999864	398315	601685	34
999987	895099	104901	403199	999861	403338	596662	33
999986	910894	089106	408161	999858	408304	591696	32
999985	926134	073866	413068	999854	413213	586787	31
999983	940858	059142	417919	999851	418068	581932	30
9.999982	7.955100	12.044900	8.422717	9.999848	8.422869	11.577131	29
999981	968889	031111	427462	999844	427618	572382	28
999980	982253	017747	432156	999841	432315	567685	27
999979	995219	004781	436800	999838	436962	563038	26
999977	8.007809	11.992191	441394	999834	441560	558440	25
999976	020045	979955	445941	999831	446110	553890	24
999975	031945	968055	450440	999827	450613	549387	23
999973	043527	956473	454893	999823	455070	544930	22
999972	054809	945191	459301	999820	459481	540519	21
999971	065806	934194	463665	999816	463849	536151	20
9.999969	8.076531	11.923469	8.467985	9.999812	8.468172	11.531828	19
999968	086997	913003	472263	999809	472454	527546	18
999966	097217	902783	476498	999805	476693	523307	17
999964	107202	892797	480693	999801	480892	519108	16
999963	116963	883037	484848	999797	485050	514950	15
999961	126510	873490	488963	999793	489170	510830	14
999959	135851	864149	493040	999790	493250	506750	13
999958	144996	855004	497078	999786	497293	502707	12
999956	153952	846048	501080	999782	501298	498702	11
999954	162727	837273	505045	999778	505267	494733	10
9.999952	8.171328	11.828672	8.508974	9.999774	8.509200	11.490800	9
999950	179763	820237	512867	999769	513098	486902	8
999948	188036	811964	516726	999765	516961	483039	7
999946	196156	803844	520551	999761	520790	479210	6
999944	204126	795874	524343	999757	524586	475414	5
999942	211953	788047	528102	999753	528349	471651	4
999940	219641	780359	531828	999748	532080	467920	3
999938	227195	772805	535523	999744	535779	464221	2
999936	234621	765379	539186	999740	539447	460553	1
999934	241921	758079	542819	999735	543084	456916	0
Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
89 Degrees.			88 Degrees.				

′	2 Degrees. Sine.	Cosine.	Tang.	Cotang.	3 Degrees. Sine.	Cosine.	Tang.
0	8.542819	9.999735	8.543084	11.456916	8.718800	9.999404	8.719396
1	546422	999731	546691	453309	721204	999398	721806
2	549995	999726	550268	449732	723595	999391	724204
3	553539	999722	553817	446183	725972	999384	726588
4	557054	999717	557336	442664	728337	999378	728959
5	560540	999713	560828	439172	730688	999371	731317
6	563999	999708	564291	435709	733027	999364	733663
7	567431	999704	567727	432273	735354	999357	735996
8	570836	999699	571137	428863	737667	999350	738317
9	574214	999694	574520	425480	739969	999343	740626
10	577566	999689	577877	422123	742259	999336	742922
11	8.580892	9.999685	8.581208	11.418792	8.744536	9.999329	8.745207
12	584193	999680	584514	415486	746802	999322	747479
13	587469	999675	587795	412205	749055	999315	749740
14	590721	999670	591051	408949	751297	999308	751989
15	593948	999665	594283	405717	753528	999301	754227
16	597152	999660	597492	402508	755747	999294	756453
17	600332	999655	600677	399323	757955	999286	758668
18	603489	999650	603839	396161	760151	999279	760872
19	606623	999645	606978	393022	762337	999272	763065
20	609734	999640	610094	389906	764511	999265	765246
21	8.612823	9.999635	8.613189	11.386811	8.766675	9.999257	8.767417
22	615891	999629	616262	383738	768828	999250	769578
23	618937	999624	619313	380687	770970	999242	771727
24	621962	999619	622343	377657	773101	999235	773866
25	624965	999614	625352	374648	775223	999227	775995
26	627948	999608	628340	371660	777333	999220	778114
27	630911	999603	631308	368692	779434	999212	780222
28	633854	999597	634256	365744	781524	999205	782320
29	636776	999592	637184	362816	783605	999197	784408
30	639680	999586	640093	359907	785675	999189	786486
31	8.642563	9.999581	8.642982	11.357018	8.787736	9.999181	8.788554
32	645428	999575	645853	354147	789787	999174	790613
33	648274	999570	648704	351296	791828	999166	792662
34	651102	999564	651537	348463	793859	999158	794701
35	653911	999558	654352	345648	795881	999150	796731
36	656702	999553	657149	342851	797894	999142	798752
37	659475	999547	659928	340072	799897	999134	800763
38	662230	999541	662689	337311	801892	999126	802765
39	664968	999535	665433	334567	803876	999118	804758
40	667689	999529	668160	331840	805852	999110	806742
41	8.670393	9.999524	8.670870	11.329130	8.807819	9.999102	8.808717
42	673080	999518	673563	326437	809777	999094	810683
43	675751	999512	676239	323761	811726	999086	812641
44	678405	999506	678900	321100	813667	999077	814589
45	681043	999500	681544	318456	815599	999069	816529
46	683665	999493	684172	315828	817522	999061	818461
47	686272	999487	686784	313216	819436	999053	820384
48	688863	999481	689381	310619	821343	999044	822298
49	691438	999475	691963	308037	823240	999036	824205
50	693998	999469	694529	305471	825130	999027	826103
51	8.696543	9.999463	8.697081	11.302919	8.827011	9.999019	8.827992
52	699073	999456	699617	300383	828884	999010	829874
53	701589	999450	702139	297861	830749	999002	831748
54	704090	999443	704646	295354	832607	998993	833613
55	706577	999437	707140	292860	834456	998984	835471
56	709049	999431	709618	290382	836297	998976	837321
57	711507	999424	712083	287917	838130	998967	839163
58	713952	999418	714534	285465	839956	998958	840998
59	716383	999411	716972	283028	841774	998950	842825
60	718800	999404	719396	280604	843585	998941	844644
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.
	87 Degrees.				86 Degrees.		

	4 Degrees.			5 Degrees.				
	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
5	9.998941	8.844644	11.155356	8.940296	9.998344	8.941952	11.058048	60
7	998932	846455	153545	941738	998333	943404	056596	59
8	998923	848260	151740	943174	998322	944852	055148	58
1	998914	850057	149943	944606	998311	946295	053705	57
4	998905	851846	148154	946034	998300	947734	052266	56
5	998896	853628	146372	947456	998289	949168	050832	55
4	998887	855403	144597	948874	998277	950597	049403	54
9	998878	857171	142829	950287	998266	952021	047979	53
1	998869	858932	141068	951696	998255	953441	046559	52
6	998860	860686	139314	953100	998243	954856	045144	51
3	998851	862433	137567	954499	998232	956267	043733	50
4	9.998841	8.864173	11.135827	8.955894	9.998220	8.957674	11.042326	49
3	998832	865906	134094	957284	998209	959075	040925	48
5	998823	867632	132368	958670	998197	960473	039527	47
5	998813	869351	130649	960052	998186	961866	038134	46
3	998804	871064	128936	961429	998174	963255	036745	45
5	998795	872770	127230	962801	998163	964639	035361	44
5	998785	874469	125531	964170	998151	966019	033981	43
3	998776	876162	123838	965534	998139	967394	032606	42
5	998766	877849	122151	966893	998128	968766	031234	41
5	998757	879529	120471	968249	998116	970133	029867	40
9	9.998747	8.881202	11.118798	8.969600	9.998104	8.971496	11.028504	39
7	998738	882869	117131	970947	998092	972855	027145	38
8	998728	884530	115470	972289	998080	974209	025791	37
8	998718	886185	113815	973628	998068	975560	024440	36
2	998708	887833	112167	974962	998056	976906	023094	35
4	998699	889476	110524	976293	998044	978248	021752	34
1	998689	891112	108888	977619	998032	979586	020414	33
1	998679	892742	107258	978941	998020	980921	019079	32
5	998669	894366	105634	980259	998008	982251	017749	31
3	998659	895984	104016	981573	997996	983577	016423	30
6	9.998649	8.897596	11.102404	8.982883	9.997984	8.984899	11.015101	29
2	998639	899203	100797	984189	997972	986217	013783	28
2	998629	900803	099197	985491	997959	987532	012468	27
7	998619	902398	097602	986789	997947	988842	011158	26
6	998609	903987	096013	988083	997935	990149	009851	25
9	998599	905570	094430	989374	997922	991451	008549	24
6	998589	907147	092853	990660	997910	992750	007250	23
7	998578	908719	091281	991943	997897	994045	005955	22
8	998568	910285	089715	993222	997885	995337	004663	21
4	998558	911846	088154	994497	997872	996624	003376	20
9	9.998548	8.913401	11.086599	8.995768	9.997860	8.997908	11.002092	19
8	998537	914951	085049	997036	997847	999188	000812	18
2	998527	916495	083505	998299	997835	9.000465	10.999535	17
0	998516	918034	081966	999560	997822	001738	998262	16
3	998506	919568	080432	9.000816	997809	003007	996993	15
1	998495	921096	078904	002069	997797	004272	995728	14
3	998485	922619	077381	003318	997784	005534	994466	13
0	998474	924136	075864	004563	997771	006792	993208	12
2	998464	925649	074351	005805	997758	008047	991953	11
9	998453	927156	072844	007044	997745	009298	990702	10
00	9.998442	8.928658	11.071342	9.008278	9.997732	9.010546	10.989454	9
37	998431	930155	069845	009510	997719	011790	988210	8
68	998421	931647	068353	010737	997706	013031	986969	7
44	998410	933134	066866	011962	997693	014268	985732	6
45	998399	934616	065384	013182	997680	015502	984498	5
31	998388	936093	063907	014400	997667	016732	983268	4
42	998377	937565	062435	015613	997654	017959	982041	3
88	998366	939032	060968	016824	997641	019183	980817	2
60	998355	940494	059506	018031	997628	020403	979597	1
06	998344	941952	058048	019235	997614	021620	978380	0
	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	85 Degrees.			84 Degrees.				

	6 Degrees.				7 Degrees.	
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.
0	9.019235	9.997614	9.021620	10.978380	9.085894	9.996751
1	020435	997601	022834	977166	086922	996735
2	021632	997588	024044	975956	087947	996720
3	022825	997574	025251	974749	088970	996704
4	024016	997561	026455	973545	089990	996688
5	025203	997547	027655	972345	091008	996673
6	026386	997534	028852	971148	092024	996657
7	027567	997520	030046	969954	093037	996641
8	028744	997507	031237	968763	094047	996625
9	029918	997493	032425	967575	095056	996610
10	031089	997480	033609	966391	096062	996594
11	9.032257	9.997466	9.034791	10.965209	9.097065	9.996578
12	033421	997452	035969	964031	098066	996562
13	034582	997439	037144	962856	099065	996546
14	035741	997425	038316	961684	100062	996530
15	036896	997411	039485	960515	101056	996514
16	038048	997397	040651	959349	102048	996498
17	039197	997383	041813	958187	103037	996482
18	040342	997369	042973	957027	104025	996465
19	041485	997355	044130	955870	105010	996449
20	042625	997341	045284	954716	105992	996433
21	9.043762	9.997327	9.046434	10.953566	9.106973	9.996417
22	044895	997313	047582	952418	107951	996400
23	046026	997299	048727	951273	108927	996384
24	047154	997285	049869	950131	109901	996368
25	048279	997271	051008	948992	110873	996351
26	049400	997257	052144	947856	111842	996335
27	050519	997242	053277	946723	112809	996318
28	051635	997228	054407	945593	113774	996302
29	052749	997214	055535	944465	114737	996285
30	053859	997199	056659	943341	115698	996269
31	9.054966	9.997185	9.057781	10.942219	9.116656	9.996252
32	056071	997170	058900	941100	117613	996235
33	057172	997156	060016	939984	118567	996219
34	058271	997141	061130	938870	119519	996202
35	059367	997127	062240	937760	120469	996185
36	060460	997112	063348	936652	121417	996168
37	061551	997098	064453	935547	122362	996151
38	062639	997083	065556	934444	123306	996134
39	063724	997068	066655	933345	124248	996117
40	064806	997053	067752	932248	125187	996100
41	9.065885	9.997039	9.068846	10.931154	9.126125	9.996083
42	066962	997024	069938	930062	127060	996066
43	068036	997009	071027	928973	127993	996049
44	069107	996994	072113	927887	128925	996032
45	070176	996979	073197	926803	129854	996015
46	071242	996964	074278	925722	130781	995998
47	072306	996949	075356	924644	131706	995980
48	073366	996934	076432	923568	132630	995963
49	074424	996919	077505	922495	133551	995946
50	075480	996904	078576	921424	134470	995928
51	9.076533	9.996889	9.079644	10.920356	9.135387	9.995911
52	077583	996874	080710	919290	136303	995894
53	078631	996858	081773	918227	137216	995876
54	079676	996843	082833	917167	138128	995859
55	080719	996828	083891	916109	139037	995841
56	081759	996812	084947	915053	139944	995823
57	082797	996797	086000	914000	140850	995806
58	083832	996782	087050	912950	141754	995788
59	084864	996766	088098	911902	142655	995771
60	085894	996751	089144	910856	143555	995753
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.
	83 Degrees.				82 Degrees.	

8 Degrees.			9 Degrees.				
Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	'
9.995753	9.147803	10.852197	9.194332	9.994620	9.199713	10.800287	60
995735	148718	851282	195129	994600	200529	799471	59
995717	149632	850368	195925	994580	201345	798655	58
995699	150544	849456	196719	994560	202159	797841	57
995681	151454	848546	197511	994540	202971	797029	56
995664	152363	847637	198302	994519	203782	796218	55
995646	153269	846731	199091	994499	204592	795408	54
995628	154174	845826	199879	994479	205400	794600	53
995610	155077	844923	200666	994459	206207	793793	52
995591	155978	844022	201451	994438	207013	792987	51
995573	156877	843123	202234	994418	207817	792183	50
9.995555	9.157775	10.842225	9.203017	9.994398	9.208619	10.791381	49
995537	158671	841329	203797	994377	209420	790580	48
995519	159565	840435	204577	994357	210220	789780	47
995501	160457	839543	205354	994336	211018	788982	46
995482	161347	838653	206131	994316	211815	788185	45
995464	162236	837764	206906	994295	212611	787389	44
995446	163123	836877	207679	994274	213405	786595	43
995427	164008	835992	208452	994254	214198	785802	42
995409	164892	835108	209222	994233	214989	785011	41
995390	165774	834226	209992	994212	215780	784220	40
9.995372	9.166654	10.833346	9.210760	9.994191	9.216568	10.783432	39
995353	167532	832468	211526	994171	217356	782644	38
995334	168409	831591	212291	994150	218142	781858	37
995316	169284	830716	213055	994129	218926	781074	36
995297	170157	829843	213818	994108	219710	780290	35
995278	171029	828971	214579	994087	220492	779508	34
995260	171899	828101	215338	994066	221272	778728	33
995241	172767	827233	216097	994045	222052	777948	32
995222	173634	826366	216854	994024	222830	777170	31
995203	174499	825501	217609	994003	223607	776393	30
9.995184	9.175362	10.824638	9.218363	9.993982	9.224382	10.775618	29
995165	176224	823776	219116	993960	225156	774844	28
995146	177084	822916	219868	993939	225929	774071	27
995127	177942	822058	220618	993918	226700	773300	26
995108	178799	821201	221367	993897	227471	772529	25
995089	179655	820345	222115	993875	228239	771761	24
995070	180508	819492	222861	993854	229007	770993	23
995051	181360	818640	223606	993832	229773	770227	22
995032	182211	817789	224349	993811	230539	769461	21
995013	183059	816941	225092	993789	231302	768698	20
994993	9.183907	10.816093	9.225833	9.993768	9.232065	10.767935	19
994974	184752	815248	226573	993746	232826	767174	18
994955	185597	814403	227311	993725	233586	766414	17
994935	186439	813561	228048	993703	234345	765655	16
994916	187280	812720	228784	993681	235103	764897	15
994896	188120	811880	229518	993660	235859	764141	14
94877	188958	811042	230252	993638	236614	763386	13
94857	189794	810206	230984	993616	237368	762632	12
94838	190629	809371	231714	993594	238120	761880	11
94818	191462	808538	232444	993572	238872	761128	10
4798	9.192294	10.807706	9.233172	9.993550	9.239622	10.760378	9
4779	193124	806876	233899	993528	240371	759629	8
4759	193953	806047	234625	993506	241118	758882	7
4739	194780	805220	235349	993484	241865	758135	6
719	195606	804394	236073	993462	242610	757390	5
700	196430	803570	236795	993440	243354	756646	4
680	197253	802747	237515	993418	244097	755903	3
660	198074	801926	238235	993396	244839	755161	2
40	198894	801106	238953	993374	245579	754421	1
0	199713	800287	239670	993351	246319	753681	0
	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	'
rees.			80 Degrees.				

′	2 Degrees.				3 Degrees.		
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.
0	8.542819	9.999735	8.543084	11.456916	8.718800	9.999404	8.719396
1	546422	999731	546691	453309	721204	999398	721806
2	549995	999726	550268	449732	723595	999391	724204
3	553539	999722	553817	446183	725972	999384	726588
4	557054	999717	557336	442664	728337	999378	728959
5	560540	999713	560828	439172	730688	999371	731317
6	563999	999708	564291	435709	733027	999364	733663
7	567431	999704	567727	432273	735354	999357	735996
8	570836	999699	571137	428863	737667	999350	738317
9	574214	999694	574520	425480	739969	999343	740626
10	577566	999689	577877	422123	742259	999336	742922
11	8.580892	9.999685	8.581208	11.418792	8.744536	9.999329	8.745207
12	584193	999680	584514	415486	746802	999322	747479
13	587469	999675	587795	412205	749055	999315	749740
14	590721	999670	591051	408949	751297	999308	751989
15	593948	999665	594283	405717	753528	999301	754227
16	597152	999660	597492	402508	755747	999294	756453
17	600332	999655	600677	399323	757955	999286	758668
18	603489	999650	603839	396161	760151	999279	760872
19	606623	999645	606978	393022	762337	999272	763065
20	609734	999640	610094	389906	764511	999265	765246
21	8.612823	9.999635	8.613189	11.386811	8.766675	9.999257	8.767417
22	615891	999629	616262	383738	768828	999250	769578
23	618937	999624	619313	380687	770970	999242	771727
24	621962	999619	622343	377657	773101	999235	773866
25	624965	999614	625352	374648	775223	999227	775995
26	627948	999608	628340	371660	777333	999220	778114
27	630911	999603	631308	368692	779434	999212	780222
28	633854	999597	634256	365744	781524	999205	782320
29	636776	999592	637184	362816	783605	999197	784408
30	639680	999586	640093	359907	785675	999189	786486
31	8.642563	9.999581	8.642982	11.357018	8.787736	9.999181	8.788554
32	645428	999575	645853	354147	789787	999174	790613
33	648274	999570	648704	351296	791828	999166	792662
34	651102	999564	651537	348463	793859	999158	794701
35	653911	999558	654352	345648	795881	999150	796731
36	656702	999553	657149	342851	797894	999142	798752
37	659475	999547	659928	340072	799897	999134	800763
38	662230	999541	662689	337311	801892	999126	802766
39	664968	999535	665433	334567	803876	999118	804758
40	667689	999529	668160	331840	805852	999110	806742
41	8.670393	9.999524	8.670870	11.329130	8.807819	9.999102	8.808717
42	673080	999518	673563	326437	809777	999094	810683
43	675751	999512	676239	323761	811726	999086	812641
44	678405	999506	678900	321100	813667	999077	814589
45	681043	999500	681544	318456	815599	999069	816529
46	683665	999493	684172	315828	817522	999061	818461
47	686272	999487	686784	313216	819436	999053	820384
48	688863	999481	689381	310619	821343	999044	822298
49	691438	999475	691963	308037	823240	999036	824205
50	693998	999469	694529	305471	825130	999027	826103
51	8.696543	9.999463	8.697081	11.302919	8.827011	9.999019	8.827992
52	699073	999456	699617	300383	828884	999010	829874
53	701589	999450	702139	297861	830749	999002	831748
54	704090	999443	704646	295354	832607	998993	833613
55	706577	999437	707140	292860	834456	998984	835471
56	709049	999431	709618	290382	836297	998976	837321
57	711507	999424	712083	287917	838130	998967	839163
58	713952	999418	714534	285465	839956	998958	840998
59	716383	999411	716972	283028	841774	998950	842825
60	718800	999404	719396	280604	843585	998941	844644
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.
	87 Degrees.				86 Degrees.		

	4 Degrees.			5 Degrees.				
	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
5	9.998941	8.844644	11.155356	8.940296	9.998344	8.941952	11.058048	60
7	998932	846455	153545	941738	998333	943404	056596	59
8	998923	848260	151740	943174	998322	944852	055148	58
1	998914	850057	149943	944606	998311	946295	053705	57
1	998905	851846	148154	946034	998300	947734	052266	56
6	998896	853628	146372	947456	998289	949168	050832	55
1	998887	855403	144597	948874	998277	950597	049403	54
9	998878	857171	142829	950287	998266	952021	047979	53
1	998869	858932	141068	951696	998255	953441	046559	52
6	998860	860686	139314	953100	998243	954856	045144	51
9	998851	862433	137567	954499	998232	956267	043733	50
4	9.998841	8.864173	11.135827	8.955894	9.998220	8.957674	11.042326	49
3	998832	865906	134094	957284	998209	959075	040925	48
5	998823	867632	132368	958670	998197	960473	039527	47
5	998813	869351	130649	960052	998186	961866	038134	46
3	998804	871064	128936	961429	998174	963255	036745	45
5	998795	872770	127230	962801	998163	964639	035361	44
5	998785	874469	125531	964170	998151	966019	033981	43
3	998776	876162	123838	965534	998139	967394	032606	42
5	998766	877849	122151	966893	998128	968766	031234	41
5	998757	879529	120471	968249	998116	970133	029867	40
9	9.998747	8.881202	11.118798	8.969600	9.998104	8.971496	11.028504	39
7	998738	882869	117131	970947	998092	972855	027145	38
3	998728	884530	115470	972289	998080	974209	025791	37
3	998718	886185	113815	973628	998068	975560	024440	36
2	998708	887833	112167	974962	998056	976906	023094	35
4	998699	889476	110524	976293	998044	978248	021752	34
1	998689	891112	108888	977619	998032	979586	020414	33
1	998679	892742	107258	978941	998020	980921	019079	32
5	998669	894366	105634	980259	998008	982251	017749	31
3	998659	895984	104016	981573	997996	983577	016423	30
6	9.998649	8.897596	11.102404	8.982883	9.997984	8.984899	11.015101	29
2	998639	899203	100797	984189	997972	986217	013783	28
2	998629	900803	099197	985491	997959	987532	012468	27
7	998619	902398	097602	986789	997947	988842	011158	26
6	998609	903987	096013	988083	997935	990149	009851	25
9	998599	905570	094430	989374	997922	991451	008549	24
6	998589	907147	092853	990660	997910	992750	007250	23
7	998578	908719	091281	991943	997897	994045	005955	22
3	998568	910285	089715	993222	997885	995337	004663	21
4	998558	911846	088154	994497	997872	996624	003376	20
9	9.998548	8.913401	11.086599	8.995768	9.997860	8.997908	11.002092	19
8	998537	914951	085049	997036	997847	999188	000812	18
2	998527	916495	083505	998299	997835	9.000465	10.999535	17
0	998516	918034	081966	999560	997822	001738	998262	16
3	998506	919568	080432	9.000816	997809	003007	996993	15
1	998495	921096	078904	002069	997797	004272	995728	14
3	998485	922619	077381	003318	997784	005534	994466	13
0	998474	924136	075864	004563	997771	006792	993208	12
2	998464	925649	074351	005805	997758	008047	991953	11
9	998453	927156	072844	007044	997745	009298	990702	10
0	9.998442	8.928658	11.071342	9.008278	9.997732	9.010546	10.989454	9
7	998431	930155	069845	009510	997719	011790	988210	8
8	998421	931647	068353	010737	997706	013031	986969	7
4	998410	933134	066866	011962	997693	014268	985732	6
5	998399	934616	065384	013182	997680	015502	984498	5
1	998388	936093	063907	014400	997667	016732	983268	4
2	998377	937565	062435	015613	997654	017959	982041	3
8	998366	939032	060968	016824	997641	019183	980817	2
0	998355	940494	059506	018031	997628	020403	979597	1
6	998344	941952	058048	019235	997614	021620	978380	0
	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	85 Degrees.			84 Degrees.				

	6 Degrees.				7 Degrees.		
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.
0	9.019235	9.997614	9.021620	10.978380	9.085894	9.996751	[illegible]
1	020435	997601	022834	977166	086922	996735	09017
2	021632	997588	024044	975956	087947	996720	[illegible]
3	022825	997574	025251	974749	088970	996704	[illegible]
4	024016	997561	026455	973545	089990	996688	[illegible]
5	025203	997547	027655	972345	091008	996673	09436
6	026386	997534	028852	971148	092024	996657	[illegible]
7	027567	997520	030046	969954	093037	996641	[illegible]
8	028744	997507	031237	968763	094047	996625	09742
9	029918	997493	032425	967575	095056	996610	[illegible]
10	031089	997480	033609	966391	096062	996594	[illegible]
11	9.032257	9.997466	9.034791	10.965209	9.097065	9.996578	9.10042
12	033421	997452	035969	964031	098066	996562	[illegible]
13	034582	997439	037144	962856	099065	996546	[illegible]
14	035741	997425	038316	961684	100062	996530	[illegible]
15	036896	997411	039485	960515	101056	996514	[illegible]
16	038048	997397	040651	959349	102048	996498	[illegible]
17	039197	997383	041813	958187	103037	996482	[illegible]
18	040342	997369	042973	957027	104025	996465	[illegible]
19	041485	997355	044130	955870	105010	996449	[illegible]
20	042625	997341	045284	954716	105992	996433	[illegible]
21	9.043762	9.997327	9.046434	10.953566	9.106973	9.996417	9.11055
22	044895	997313	047582	952418	107951	996400	11155
23	046026	997299	048727	951273	108927	996384	11254
24	047154	997285	049869	950131	109901	996368	11353
25	048279	997271	051008	948992	110873	996351	11452
26	049400	997257	052144	947856	111842	996335	[illegible]
27	050519	997242	053277	946723	112809	996318	11649
28	051635	997228	054407	945593	113774	996302	11747
29	052749	997214	055535	944465	114737	996285	11845
30	053859	997199	056659	943341	115698	996269	11942
31	9.054966	9.997185	9.057781	10.942219	9.116656	9.996252	9.12040
32	056071	997170	058900	941100	117613	996235	12137
33	057172	997156	060016	939984	118567	996219	[illegible]
34	058271	997141	061130	938870	119519	996202	[illegible]
35	059367	997127	062240	937760	120469	996185	[illegible]
36	060460	997112	063348	936652	121417	996168	[illegible]
37	061551	997098	064453	935547	122362	996151	[illegible]
38	062639	997083	065556	934444	123306	996134	12717
39	063724	997068	066655	933345	124248	996117	12813
40	064806	997053	067752	932248	125187	996100	[illegible]
41	9.065885	9.997039	9.068846	10.931154	9.126125	9.996083	9.13004
42	066962	997024	069938	930062	127060	996066	[illegible]
43	068036	997009	071027	928973	127993	996049	13194
44	069107	996994	072113	927887	128925	996032	[illegible]
45	070176	996979	073197	926803	129854	996015	[illegible]
46	071242	996964	074278	925722	130781	995998	13478
47	072306	996949	075356	924644	131706	995980	13572
48	073366	996934	076432	923568	132630	995963	13666
49	074424	996919	077505	922495	133551	995946	[illegible]
50	075480	996904	078576	921424	134470	995928	[illegible]
51	9.076533	9.996889	9.079644	10.920356	9.135387	9.995911	9.139476
52	077583	996874	080710	919290	136303	995894	140409
53	078631	996858	081773	918227	137216	995876	141340
54	079676	996843	082833	917167	138128	995859	[illegible]
55	080719	996828	083891	916109	139037	995841	143196
56	081759	996812	084947	915053	139944	995823	144121
57	082797	996797	086000	914000	140850	995806	145044
58	083832	996782	087050	912950	141754	995788	[illegible]
59	084864	996766	088098	911902	142655	995771	[illegible]
60	085894	996751	089144	910856	143555	995753	[illegible]
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.
	83 Degrees.				82 Degrees.		

8 Degrees.			9 Degrees.				
Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.995753	9.147803	10.852197	9.194332	9.994620	9.199713	10.800287	60
995735	148718	851282	195129	994600	200529	799471	59
995717	149632	850368	195925	994580	201345	798655	58
995699	150544	849456	196719	994560	202159	797841	57
995681	151454	848546	197511	994540	202971	797029	56
995664	152363	847637	198302	994519	203782	796218	55
995646	153269	846731	199091	994499	204592	795408	54
995628	154174	845826	199879	994479	205400	794600	53
995610	155077	844923	200666	994459	206207	793793	52
995591	155978	844022	201451	994438	207013	792987	51
995573	156877	843123	202234	994418	207817	792183	50
9.995555	9.157775	10.842225	9.203017	9.994398	9.208619	10.791381	49
995537	158671	841329	203797	994377	209420	790580	48
995519	159565	840435	204577	994357	210220	789780	47
995501	160457	839543	205354	994336	211018	788982	46
995482	161347	838653	206131	994316	211815	788185	45
995464	162236	837764	206906	994295	212611	787389	44
995446	163123	836877	207679	994274	213405	786595	43
995427	164008	835992	208452	994254	214198	785802	42
995409	164892	835108	209222	994233	214989	785011	41
995390	165774	834226	209992	994212	215780	784220	40
9.995372	9.166654	10.833346	9.210760	9.994191	9.216568	10.783432	39
995353	167532	832468	211526	994171	217356	782644	38
995334	168409	831591	212291	994150	218142	781858	37
995316	169284	830716	213055	994129	218926	781074	36
995297	170157	829843	213818	994108	219710	780290	35
995278	171029	828971	214579	994087	220492	779508	34
995260	171899	828101	215338	994066	221272	778728	33
995241	172767	827233	216097	994045	222052	777948	32
995222	173634	826366	216854	994024	222830	777170	31
995203	174499	825501	217609	994003	223607	776393	30
995184	9.175362	10.824638	9.218363	9.993982	9.224382	10.775618	29
95165	176224	823776	219116	993960	225156	774844	28
95146	177084	822916	219868	993939	225929	774071	27
95127	177942	822058	220618	993918	226700	773300	26
95108	178799	821201	221367	993897	227471	772529	25
95089	179655	820345	222115	993875	228239	771761	24
95070	180508	819492	222861	993854	229007	770993	23
95051	181360	818640	223606	993832	229773	770227	22
95032	182211	817789	224349	993811	230539	769461	21
95013	183059	816941	225092	993789	231302	768698	20
4993	9.183907	10.816093	9.225833	9.993768	9.232065	10.767935	19
4974	184752	815248	226573	993746	232826	767174	18
4955	185597	814403	227311	993725	233586	766414	17
4935	186439	813561	228048	993703	234345	765655	16
4916	187280	812720	228784	993681	235103	764897	15
4896	188120	811880	229518	993660	235859	764141	14
877	188958	811042	230252	993638	236614	763386	13
857	189794	810206	230984	993616	237368	762632	12
838	190629	809371	231714	993594	238120	761880	11
818	191462	808538	232444	993572	238872	761128	10
798	9.192294	10.807706	9.233172	9.993550	9.239622	10.760378	9
779	193124	806876	233899	993528	240371	759629	8
59	193953	806047	234625	993506	241118	758882	7
39	194780	805220	235349	993484	241865	758135	6
19	195606	804394	236073	993462	242610	757390	5
00	196430	803570	236795	993440	243354	756646	4
80	197253	802747	237515	993418	244097	755903	3
60	198074	801926	238235	993396	244839	755161	2
0	198894	801106	238953	993374	245579	754421	1
[illegible]	199713	800287	239670	993351	246319	753681	0
	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
rees.			80 Degrees.				

	10 Degrees.				11 Degrees.		
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.
0	9.239670	9.993351	9.246319	10.753681	9.280599	9.991947	9.288652
1	240386	993329	247057	752943	281248	991922	289326
2	241101	993307	247794	752206	281897	991897	289998
3	241814	993284	248530	751470	282544	991873	290671
4	242526	993262	249264	750736	283190	991848	291342
5	243237	993240	249998	750002	283836	991823	292013
6	243947	993217	250730	749270	284480	991799	292682
7	244656	993195	251461	748539	285124	991774	293350
8	245363	993172	252191	747809	285766	991749	294017
9	246069	993149	252920	747080	286408	991724	294684
10	246775	993127	253648	746352	287048	991699	295349
11	9.247478	9.993104	9.254374	10.745626	9.287687	9.991674	9.296013
12	248181	993081	255100	744900	288326	991649	296677
13	248883	993059	255824	744176	288964	991624	297339
14	249583	993036	256547	743453	289600	991599	298001
15	250282	993013	257269	742731	290236	991574	298662
16	250980	992990	257990	742010	290870	991549	299322
17	251677	992967	258710	741290	291504	991524	299980
18	252373	992944	259429	740571	292137	991498	300638
19	253067	992921	260146	739854	292768	991473	301295
20	253761	992898	260863	739137	293399	991448	301951
21	9.254453	9.992875	9.261578	10.738422	9.294029	9.991422	9.302607
22	255144	992852	262292	737708	294658	991397	303261
23	255834	992829	263005	736995	295286	991372	303914
24	256523	992806	263717	736283	295913	991346	304567
25	257211	992783	264428	735572	296539	991321	305218
26	257898	992759	265138	734862	297164	991295	305869
27	258583	992736	265847	734153	297788	991270	306519
28	259268	992713	266555	733445	298412	991244	307168
29	259951	992690	267261	732739	299034	991218	307815
30	260633	992666	267967	732033	299655	991193	308463
31	9.261314	9.992643	9.268671	10.731329	9.300276	9.991167	9.309109
32	261994	992619	269375	730625	300895	991141	309754
33	262673	992596	270077	729923	301514	991115	310398
34	263351	992572	270779	729221	302132	991090	311042
35	264027	992549	271479	728521	302748	991064	311685
36	264703	992525	272178	727822	303364	991038	312327
37	265377	992501	272876	727124	303979	991012	312967
38	266051	992478	273573	726427	304593	990986	313608
39	266723	992454	274269	725731	305207	990960	314247
40	267395	992430	274964	725036	305819	990934	314885
41	9.268065	9.992406	9.275658	10.724342	9.306430	9.990908	9.315523
42	268734	992382	276351	723649	307041	990882	316159
43	269402	992359	277043	722957	307650	990855	316795
44	270069	992335	277734	722266	308259	990829	317430
45	270735	992311	278424	721576	308867	990803	318064
46	271400	992287	279113	720887	309474	990777	318697
47	272064	992263	279801	720199	310080	990750	319329
48	272726	992239	280488	719512	310685	990724	319961
49	273388	992214	281174	718826	311289	990697	320592
50	274049	992190	281858	718142	311893	990671	321222
51	9.274708	9.992166	9.282542	10.717458	9.312495	9.990644	9.321851
52	275367	992142	283225	716775	313097	990618	322479
53	276024	992118	283907	716093	313698	990591	323106
54	276681	992093	284588	715412	314297	990565	323733
55	277337	992069	285268	714732	314897	990538	324358
56	277991	992044	285947	714053	315495	990511	324983
57	278645	992020	286624	713376	316092	990485	325607
58	279297	991996	287301	712699	316689	990458	326231
59	279948	991971	287977	712023	317284	990431	[illegible]
60	280599	991947	288652	711348	317879	990404	[illegible]
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.
	79 Degrees.				78 Degrees.		

	12 Degrees.			13 Degrees.				
	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
[illegible]	9.990404	9.327474	10.672526	9.352088	9.988724	9.363364	10.636636	60
3	990378	328095	671905	352635	988695	363940	636060	59
6	990351	328715	671285	353181	988666	364515	635485	58
3	990324	329334	670666	353726	988636	365090	634910	57
9	990297	329953	670047	354271	988607	365664	634336	56
0	990270	330570	669430	354815	988578	366237	633763	55
0	990243	331187	668813	355358	988548	366810	633190	54
9	990215	331803	668197	355901	988519	367382	632618	53
7	990188	332418	667582	356443	988489	367953	632047	52
4	990161	333033	666967	356984	988460	368524	631476	51
0	990134	333646	666354	357524	988430	369094	630906	50
6	9.990107	9.334259	10.665741	9.358064	9.988401	9.369663	10.630337	49
0	990079	334871	665129	358603	988371	370232	629768	48
4	990052	335482	664518	359141	988342	370799	629201	47
7	990025	336093	663907	359678	988312	371367	628633	46
0	989997	336702	663298	360215	988282	371933	628067	45
1	989970	337311	662689	360752	988252	372499	627501	44
2	989942	337919	662081	361287	988223	373064	626936	43
2	989915	338527	661473	361822	988193	373629	626371	42
1	989887	339133	660867	362356	988163	374193	625807	41
9	989860	339739	660261	362889	988133	374756	625244	40
6	9.989832	9.340344	10.659656	9.363422	9.988103	9.375319	10.624681	39
3	989804	340948	659052	363954	988073	375881	624119	38
9	989777	341552	658448	364485	988043	376442	623558	37
3	989749	342155	657845	365016	988013	377003	622997	36
8	989721	342757	657243	365546	987983	377563	622437	35
1	989693	343358	656642	366075	987953	378122	621878	34
4	989665	343958	656042	366604	987922	378681	621319	33
5	989637	344558	655442	367131	987892	379239	620761	32
6	989609	345157	654843	367659	987862	379797	620203	31
7	989582	345755	654245	368185	987832	380354	619646	30
6	9.989553	9.346353	10.653647	9.368711	9.987801	9.380910	10.619090	29
5	989525	346949	653051	369236	987771	381466	618534	28
3	989497	347545	652455	369761	987740	382020	617980	27
0	989469	348141	651859	370285	987710	382575	617425	26
6	989441	348735	651265	370808	987679	383129	616871	25
2	989413	349329	650671	371330	987649	383682	616318	24
7	989385	349922	650078	371852	987618	384234	615766	23
1	989356	350514	649486	372373	987588	384786	615214	22
4	989328	351106	648894	372894	987557	385337	614663	21
6	989300	351697	648303	373414	987526	385888	614112	20
8	9.989271	9.352287	10.647713	9.373933	9.987496	9.386438	10.613562	19
9	989243	352876	647124	374452	987465	386987	613013	18
9	989214	353465	646535	374970	987434	387536	612464	17
9	989186	354053	645947	375487	987403	388084	611916	16
7	989157	354640	645360	376003	987372	388631	611369	15
5	989128	355227	644773	376519	987341	389178	610822	14
2	989100	355813	644187	377035	987310	389724	610276	13
9	989071	356398	643602	377549	987279	390270	609730	12
4	989042	356982	643018	378063	987248	390815	609185	11
9	989014	357566	642434	378577	987217	391360	608640	10
4	9.988985	9.358149	10.641851	9.379089	9.987186	9.391903	10.608097	9
7	988956	358731	641269	379601	987155	392447	607553	8
0	988927	359313	640687	380113	987124	392989	607011	7
2	988898	359893	640107	380624	987092	393531	606469	6
3	988869	360474	639526	381134	987061	394073	605927	5
3	988840	361053	638947	381643	987030	394614	605386	4
3	988811	361632	638368	382152	986998	395154	604846	3
2	988782	362210	637790	382661	986967	395694	604306	2
0	988753	362787	637213	383168	986936	396233	603767	1
8	988724	363364	636636	383675	986904	396771	603229	0
	Sine.	Cotang.	Tang.	Cosine.	Sine.	C[illegible]	[illegible]	
	77 Degrees.			76 Degrees.				

	10 Degrees.				11 Degrees.		
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.
0	9.239670	9.993351	9.246319	10.753681	9.280599	9.991947	9.288652
1	240386	993329	247057	752943	281248	991922	289326
2	241101	993307	247794	752206	281897	991897	289999
3	241814	993284	248530	751470	282544	991873	290671
4	242526	993262	249264	750736	283190	991848	291342
5	243237	993240	249998	750002	283836	991823	292013
6	243947	993217	250730	749270	284480	991799	292682
7	244656	993195	251461	748539	285124	991774	293350
8	245363	993172	252191	747809	285766	991749	294017
9	246069	993149	252920	747080	286408	991724	294684
10	246775	993127	253648	746352	287048	991699	295349
11	9.247478	9.993104	9.254374	10.745626	9.287687	9.991674	9.296013
12	248181	993081	255100	744900	288326	991649	296677
13	248883	993059	255824	744176	288964	991624	297339
14	249583	993036	256547	743453	289600	991599	298001
15	250282	993013	257269	742731	290236	991574	298662
16	250980	992990	257990	742010	290870	991549	299322
17	251677	992967	258710	741290	291504	991524	299980
18	252373	992944	259429	740571	292137	991498	300638
19	253067	992921	260146	739854	292768	991473	301295
20	253761	992898	260863	739137	293399	991448	301951
21	9.254453	9.992875	9.261578	10.738422	9.294029	9.991422	9.302607
22	255144	992852	262292	737708	294658	991397	303261
23	255834	992829	263005	736995	295286	991372	303914
24	256523	992806	263717	736283	295913	991346	304567
25	257211	992783	264428	735572	296539	991321	305218
26	257898	992759	265138	734862	297164	991295	305869
27	258583	992736	265847	734153	297788	991270	306519
28	259268	992713	266555	733445	298412	991244	307168
29	259951	992690	267261	732739	299034	991218	307815
30	260633	992666	267967	732033	299655	991193	308463
31	9.261314	9.992643	9.268671	10.731329	9.300276	9.991167	9.309109
32	261994	992619	269375	730625	300895	991141	309754
33	262673	992596	270077	729923	301514	991115	310398
34	263351	992572	270779	729221	302132	991090	311042
35	264027	992549	271479	728521	302748	991064	311685
36	264703	992525	272178	727822	303364	991038	312327
37	265377	992501	272876	727124	303979	991012	312967
38	266051	992478	273573	726427	304593	990986	313608
39	266723	992454	274269	725731	305207	990960	314247
40	267395	992430	274964	725036	305819	990934	314885
41	9.268065	9.992406	9.275658	10.724342	9.306430	9.990908	9.315523
42	268734	992382	276351	723649	307041	990882	316159
43	269402	992359	277043	722957	307650	990855	316795
44	270069	992335	277734	722266	308259	990829	317430
45	270735	992311	278424	721576	308867	990803	318064
46	271400	992287	279113	720887	309474	990777	318697
47	272064	992263	279801	720199	310080	990750	319329
48	272726	992239	280488	719512	310685	990724	319961
49	273388	992214	281174	718826	311289	990697	320592
50	274049	992190	281858	718142	311893	990671	321222
51	9.274708	9.992166	9.282542	10.717458	9.312495	9.990644	9.321851
52	275367	992142	283225	716775	313097	990618	322479
53	276024	992118	283907	716093	313698	990591	323106
54	276681	992093	284588	715412	314297	990565	323733
55	277337	992069	285268	714732	314897	990538	324358
56	277991	992044	285947	714053	315495	990511	324983
57	278645	992020	286624	713376	316092	990485	325607
58	279297	991996	287301	712699	316689	990458	326231
59	279948	991971	287977	712023	317284	990431	326853
60	280599	991947	288652	711348	317879	990404	327475
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.
	79 Degrees.				78 Degrees.		

	12 Degrees.			13 Degrees.				
	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9	9.990404	9.327474	10.672526	9.352088	9.988724	9.363364	10.636636	60
3	990378	328095	671905	352635	988695	363940	636060	59
6	990351	328715	671285	353181	988666	364515	635485	58
8	990324	329334	670666	353726	988636	365090	634910	57
9	990297	329953	670047	354271	988607	365664	634336	56
0	990270	330570	669430	354815	988578	366237	633763	55
0	990243	331187	668813	355358	988548	366810	633190	54
9	990215	331803	668197	355901	988519	367382	632618	53
7	990188	332418	667582	356443	988489	367953	632047	52
4	990161	333033	666967	356984	988460	368524	631476	51
0	990134	333646	666354	357524	988430	369094	630906	50
6	9.990107	9.334259	10.665741	9.358064	9.988401	9.369663	10.630337	49
0	990079	334871	665129	358603	988371	370232	629768	48
4	990052	335482	664518	359141	988342	370799	629201	47
7	990025	336093	663907	359678	988312	371367	628633	46
0	989997	336702	663298	360215	988282	371933	628067	45
1	989970	337311	662689	360752	988252	372499	627501	44
2	989942	337919	662081	361287	988223	373064	626936	43
2	989915	338527	661473	361822	988193	373629	626371	42
1	989887	339133	660867	362356	988163	374193	625807	41
9	989860	339739	660261	362889	988133	374756	625244	40
6	9.989832	9.340344	10.659656	9.363422	9.988103	9.375319	10.624681	39
3	989804	340948	659052	363954	988073	375881	624119	38
9	989777	341552	658448	364485	988043	376442	623558	37
3	989749	342155	657845	365016	988013	377003	622997	36
8	989721	342757	657243	365546	987983	377563	622437	35
1	989693	343358	656642	366075	987953	378122	621878	34
4	989665	343958	656042	366604	987922	378681	621319	33
5	989637	344558	655442	367131	987892	379239	620761	32
6	989609	345157	654843	367659	987862	379797	620203	31
7	989582	345755	654245	368185	987832	380354	619646	30
6	9.989553	9.346353	10.653647	9.368711	9.987801	9.380910	10.619090	29
5	989525	346949	653051	369236	987771	381466	618534	28
3	989497	347545	652455	369761	987740	382020	617980	27
0	989469	348141	651859	370285	987710	382575	617425	26
6	989441	348735	651265	370808	987679	383129	616871	25
2	989413	349329	650671	371330	987649	383682	616318	24
7	989385	349922	650078	371852	987618	384234	615766	23
1	989356	350514	649486	372373	987588	384786	615214	22
4	989328	351106	648894	372894	987557	385337	614663	21
6	989300	351697	648303	373414	987526	385888	614112	20
8	9.989271	9.352287	10.647713	9.373933	9.987496	9.386438	10.613562	19
9	989243	352876	647124	374452	987465	386987	613013	18
9	989214	353465	646535	374970	987434	387536	612464	17
9	989186	354053	645947	375487	987403	388084	611916	16
7	989157	354640	645360	376003	987372	388631	611369	15
5	989128	355227	644773	376519	987341	389178	610822	14
2	989100	355813	644187	377035	987310	389724	610276	13
9	989071	356398	643602	377549	987279	390270	609730	12
4	989042	356982	643018	378063	987248	390815	609185	11
9	989014	357566	642434	378577	987217	391360	608640	10
4	9.988985	9.358149	10.641851	9.379089	9.987186	9.391903	10.608097	9
7	988956	358731	641269	379601	987155	392447	607553	8
0	988927	359313	640687	380113	987124	392989	607011	7
2	988898	359893	640107	380624	987092	393531	606469	6
3	988869	360474	639526	381134	987061	394073	605927	5
3	988840	361053	638947	381643	987030	394614	605386	4
3	988811	361632	638368	382152	986998	395154	604846	3
2	988782	362210	637790	382661	986967	395694	604306	2
0	988753	362787	637213	383168	986936	396233	603767	1
8	988724	363364	636636	383675	986904	396771	603229	0
	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	77 Degrees.			76 Degrees.				

	14 Degrees.				15 Degrees.			
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
0	9.383675	9.986904	9.396771	10.603229	9.412996	9.984944	9.428052	[illegible]
1	384182	986873	397309	602691	413467	984910	428557	[illegible]
2	384687	986841	397846	602154	413938	984876	429062	[illegible]
3	385192	986809	398383	601617	414408	984842	429566	[illegible]
4	385697	986778	398919	601081	414878	984808	430070	[illegible]
5	386201	986746	399455	600545	415347	984774	430573	[illegible]
6	386704	986714	399990	600010	415815	984740	431075	[illegible]
7	387207	986683	400524	599476	416283	984706	431577	[illegible]
8	387709	986651	401058	598942	416751	984672	432079	[illegible]
9	388210	986619	401591	598409	417217	984638	432580	[illegible]
10	388711	986587	402124	597876	417684	984603	433080	[illegible]
11	9.389211	9.986555	9.402656	10.597344	9.418150	9.984569	9.433580	[illegible]
12	389711	986523	403187	596813	418615	984535	434080	[illegible]
13	390210	986491	403718	596282	419079	984500	434579	[illegible]
14	390708	986459	404249	595751	419544	984466	435078	[illegible]
15	391206	986427	404778	595222	420007	984432	435576	[illegible]
16	391703	986395	405308	594692	420470	984397	436073	[illegible]
17	392199	986363	405836	594164	420933	984363	436570	[illegible]
18	392695	986331	406364	593636	421395	984328	437067	[illegible]
19	393191	986299	406892	593108	421857	984294	437563	[illegible]
20	393685	986266	407419	592581	422318	984259	438059	[illegible]
21	9.394179	9.986234	9.407945	10.592055	9.422778	9.984224	9.438554	[illegible]
22	394673	986202	408471	591529	423238	984190	439048	[illegible]
23	395166	986169	408997	591003	423697	984155	439543	[illegible]
24	395658	986137	409521	590479	424156	984120	440036	[illegible]
25	396150	986104	410045	589955	424615	984085	440529	[illegible]
26	396641	986072	410569	589431	425073	984050	441022	
27	397132	986039	411092	588908	425530	984015	441514	
28	397621	986007	411615	588385	425987	983981	442006	
29	398111	985974	412137	587863	426443	983946	442497	
30	398600	985942	412658	587342	426899	983911	442988	
31	9.399088	9.985909	9.413179	10.586821	9.427354	9.983875	9.443479	
32	399575	985876	413699	586301	427809	983840	443968	
33	400062	985843	414219	585781	428263	983805	444458	
34	400549	985811	414738	585262	428717	983770	444947	
35	401035	985778	415257	584743	429170	983735	445435	
36	401520	985745	415775	584225	429623	983700	445923	
37	402005	985712	416293	583707	430075	983664	446411	
38	402489	985679	416810	583190	430527	983629	446898	
39	402972	985646	417326	582674	430978	983594	447384	
40	403455	985613	417842	582158	431429	983558	447870	
41	9.403938	9.985580	9.418358	10.581642	9.431879	9.983523	9.448356	
42	404420	985547	418873	581127	432329	983487	448841	
43	404901	985514	419387	580613	432778	983452	449326	
44	405382	985480	419901	580099	433226	983416	449810	
45	405862	985447	420415	579585	433675	983381	450294	
46	406341	985414	420927	579073	434122	983345	450777	
47	406820	985381	421440	578560	434569	983309	451260	
48	407299	985347	421952	578048	435016	983273	451743	
49	407777	985314	422463	577537	435462	983238	452225	
50	408254	985280	422974	577026	435908	983202	452706	
51	9.408731	9.985247	9.423484	10.576516	9.436353	9.983166	9.453187	
52	409207	985213	423993	576007	436798	983130	453668	
53	409682	985180	424503	575497	437242	983094	454148	
54	410157	985146	425011	574989	437686	983058	454628	
55	410632	985113	425519	574481	438129	983022	455107	
56	411106	985079	426027	573973	438572	982986	455586	
57	411579	985045	426534	573466	439014	982950	456064	
58	412052	985011	427041	572959	439456	982914	456542	
59	412524	984978	427547	572453	439897	982878	457019	
60	412996	984944	428052	571948	440338	982842	45749	
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotan	
	75 Degrees.				74 Deg			

rees.		17 Degrees.				
Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.457496	10.542504	9.465935	9.980596	9.485339	10.514661	60
457973	542027	466348	980558	485791	514209	59
458449	541551	466761	980519	486242	513758	58
458925	541075	467173	980480	486693	513307	57
459400	540600	467585	980442	487143	512857	56
459875	540125	467996	980403	487593	512407	55
460349	539651	468407	980364	488043	511957	54
460823	539177	468817	980325	488492	511508	53
461297	538703	469227	980286	488941	511059	52
461770	538230	469637	980247	489390	510610	51
462242	537758	470046	980208	489838	510162	50
9.462714	10.537286	9.470455	9.980169	9.490286	10.509714	49
463186	536814	470863	980130	490733	509267	48
463658	536342	471271	980091	491180	508820	47
464128	535872	471679	980052	491627	508373	46
464599	535401	472086	980012	492073	507927	45
465069	534931	472492	979973	492519	507481	44
465539	534461	472898	979934	492965	507035	43
466008	533992	473304	979895	493410	506590	42
466476	533524	473710	979855	493854	506146	41
466945	533055	474115	979816	494299	505701	40
9.467413	10.532587	9.474519	9.979776	9.494743	10.505257	39
467880	532120	474923	979737	495186	504814	38
468347	531653	475327	979697	495630	504370	37
468814	531186	475730	979658	496073	503927	36
469280	530720	476133	979618	496515	503485	35
469746	530254	476536	979579	496957	503043	34
470211	529789	476938	979539	497399	502601	33
470676	529324	477340	979499	497841	502159	32
471141	528859	477741	979459	498282	501718	31
471605	528395	478142	979420	498722	501278	30
9.472068	10.527932	9.478542	9.979380	9.499163	10.500837	29
472532	527468	478942	979340	499603	500397	28
472995	527005	479342	979300	500042	499958	27
473457	526543	479741	979260	500481	499519	26
473919	526081	480140	979220	500920	499080	25
474381	525619	480539	979180	501359	498641	24
474842	525158	480937	979140	501797	498203	23
475303	524697	481334	979100	502235	497765	22
475763	524237	481731	979059	502672	497328	21
476223	523777	482128	979019	503109	496891	20
9.476683	10.523317	9.482525	9.978979	9.503546	10.496454	19
477142	522858	482921	978939	503982	496018	18
477601	522399	483316	978898	504418	495582	17
478059	521941	483712	978858	504854	495146	16
478517	521483	484107	978817	505289	494711	15
478975	521025	484501	978777	505724	494276	14
479432	520568	484895	978737	506159	493841	13
479889	520111	485289	978696	506593	493407	12
480345	519655	485682	978655	507027	492973	11
480801	519199	486075	978615	507460	492540	10
9.481257	10.518743	9.486467	9.978574	9.507893	10.492107	9
481712	518288	486860	978533	508326	491674	8
482167	517833	487251	978493	508759	491241	7
482621	517379	487643	978452	509191	490809	6
483075	516925	488034	978411	509622	490378	5
483529	516471	488424	978370	510054	489946	4
483982	516018	488814	978329	510485	489515	3
484435	515565	489204	978288	510916	489084	2
484887	515113	489593	978247	511346	488654	1
485339	514661	489982	978206	511776	488224	0
Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
ees.		72 Degrees.				

	18 Degrees.				19 Degrees.		
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.
0	9.489982	9.978206	9.511776	10.488224	9.512642	9.975670	9.536972
1	490371	978165	512206	487794	513009	975627	537382
2	490759	978124	512635	487365	513375	975583	537792
3	491147	978083	513064	486936	513741	975539	538202
4	491535	978042	513493	486507	514107	975496	538611
5	491922	978001	513921	486079	514472	975452	539020
6	492308	977959	514349	485651	514837	975408	539429
7	492695	977918	514777	485223	515202	975365	539837
8	493081	977877	515204	484796	515566	975321	540245
9	493466	977835	515631	484369	515930	975277	540653
10	493851	977794	516057	483943	516294	975233	541061
11	9.494236	9.977752	9.516484	10.483516	9.516657	9.975189	9.541468
12	494621	977711	516910	483090	517020	975145	541875
13	495005	977669	517335	482665	517382	975101	542281
14	495388	977628	517761	482239	517745	975057	542688
15	495772	977586	518185	481815	518107	975013	543094
16	496154	977544	518610	481390	518468	974969	543499
17	496537	977503	519034	480966	518829	974925	543905
18	496919	977461	519458	480542	519190	974880	544310
19	497301	977419	519882	480118	519551	974836	544715
20	497682	977377	520305	479695	519911	974792	545119
21	9.498064	9.977335	9.520728	10.479272	9.520271	9.974748	9.545524
22	498444	977293	521151	478849	520631	974703	545928
23	498825	977251	521573	478427	520990	974659	546331
24	499204	977209	521995	478005	521349	974614	546735
25	499584	977167	522417	477583	521707	974570	547138
26	499963	977125	522838	477162	522066	974525	547540
27	500342	977083	523259	476741	522424	974481	547943
28	500721	977041	523680	476320	522781	974436	548345
29	501099	976999	524100	475900	523138	974391	548747
30	501476	976957	524520	475480	523495	974347	549149
31	9.501854	9.976914	9.524939	10.475061	9.523852	9.974302	9.549550
32	502231	976872	525359	474641	524208	974257	549951
33	502607	976830	525778	474222	524564	974212	550352
34	502984	976787	526197	473803	524920	974167	550752
35	503360	976745	526615	473385	525275	974122	551152
36	503735	976702	527033	472967	525630	974077	551552
37	504110	976660	527451	472549	525984	974032	551952
38	504485	976617	527868	472132	526339	973987	552351
39	504860	976574	528285	471715	526693	973942	552750
40	505234	976532	528702	471298	527046	973897	553149
41	9.505608	9.976489	9.529119	10.470881	9.527400	9.973852	9.553548
42	505981	976446	529535	470465	527753	973807	553946
43	506354	976404	529950	470050	528105	973761	554344
44	506727	976361	530366	469634	528458	973716	554741
45	507099	976318	530781	469219	528810	973671	555139
46	507471	976275	531196	468804	529161	973625	555536
47	507843	976232	531611	468389	529513	973580	555933
48	508214	976189	532025	467975	529864	973535	556329
49	508585	976146	532439	467561	530215	973489	556725
50	508956	976103	532853	467147	530565	973444	557121
51	9.509326	9.976060	9.533266	10.466734	9.530915	9.973398	9.557517
52	509696	976017	533679	466321	531265	973352	557913
53	510065	975974	534092	465908	531614	973307	558308
54	510434	975930	534504	465496	531963	973261	558702
55	510803	975887	534916	465084	532312	973215	559097
56	511172	975844	535328	464672	532661	973169	559491
57	511540	975800	535739	464261	533009	973124	559885
58	511907	975757	536150	463850	533357	973078	560279
59	512275	975714	536561	463439	533704	973032	560673
60	512642	975670	536972	463028	534052	972986	[illegible]
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.
	71 Degrees.				70 Degrees.		

	grees.		21 Degrees.				
	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
6	9.561066	10.438934	9.554329	9.970152	9.584177	10.415823	60
0	561459	438541	554658	970103	584555	415445	59
4	561851	438149	554987	970055	584932	415068	58
3	562244	437756	555315	970006	585309	414691	57
2	562636	437364	555643	969957	585686	414314	56
5	563028	436972	555971	969909	586062	413938	55
9	563419	436581	556299	969860	586439	413561	54
3	563811	436189	556626	969811	586815	413185	53
7	564202	435798	556953	969762	587190	412810	52
0	564592	435408	557280	969714	587566	412434	51
4	564983	435017	557606	969665	587941	412059	50
3	9.565373	10.434627	9.557932	9.969616	9.588316	10.411684	49
1	565763	434237	558258	969567	588691	411309	48
5	566153	433847	558583	969518	589066	410934	47
8	566542	433458	558909	969469	589440	410560	46
1	566932	433068	559234	969420	589814	410186	45
5	567320	432680	559558	969370	590188	409812	44
8	567709	432291	559883	969321	590562	409438	43
1	568098	431902	560207	969272	590935	409065	42
5	568486	431514	560531	969223	591308	408692	41
8	568873	431127	560855	969173	591681	408319	40
1	9.569261	10.430739	9.561178	9.969124	9.592054	10.407946	39
4	569648	430352	561501	969075	592426	407574	38
7	570035	429965	561824	969025	592798	407202	37
0	570422	429578	562146	968976	593171	406829	36
3	570809	429191	562468	968926	593542	406458	35
6	571195	428805	562790	968877	593914	406086	34
9	571581	428419	563112	968827	594285	405715	33
2	571967	428033	563433	968777	594656	405344	32
5	572352	427648	563755	968728	595027	404973	31
8	572738	427262	564075	968678	595398	404602	30
0	9.573123	10.426877	9.564396	9.968628	9.595768	10.404232	29
3	573507	426493	564716	968578	596138	403862	28
6	573892	426108	565036	968528	596508	403492	27
8	574276	425724	565356	968479	596878	403122	26
1	574660	425340	565676	968429	597247	402753	25
3	575044	424956	565995	968379	597616	402384	24
6	575427	424573	566314	968329	597985	402015	23
8	575810	424190	566632	968278	598354	401646	22
1	576193	423807	566951	968228	598722	401278	21
3	576576	423424	567269	968178	599091	400909	20
6	9.576959	10.423041	9.567587	9.968128	9.599459	10.400541	19
8	577341	422659	567904	968078	599827	400173	18
0	577723	422277	568222	968027	600194	399806	17
2	578104	421896	568539	967977	600562	399438	16
4	578486	421514	568856	967927	600929	399071	15
7	578867	421133	569172	967876	601296	398704	14
9	579248	420752	569488	967826	601662	398338	13
1	579629	420371	569804	967775	602029	397971	12
3	580009	419991	570120	967725	602395	397605	11
5	580389	419611	570435	967674	602761	397239	10
6	9.580769	10.419231	9.570751	9.967624	9.603127	10.396873	9
8	581149	418851	571066	967573	603493	396507	8
0	581528	418472	571380	967522	603858	396142	7
2	581907	418093	571695	967471	604223	395777	6
4	582286	417714	572009	967421	604588	395412	5
5	582665	417335	572323	967370	604953	395047	4
7	583043	416957	572636	967319	605317	394683	3
9	583422	416578	572950	967268	605682	394318	2
0	583800	416200	573263	967217	606046	393954	1
2	584177	415823	573575	967166	606410	393590	0
	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	grees.		68 Degrees.				

	22 Degrees.				23 Degrees.				
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
0	9.573575	9.967166	9.606410	10.393590	9.591878	9.964026	9.627852	[illegible]	[illegible]
1	573888	967115	606773	393227	592176	963972	628203	[illegible]	[illegible]
2	574200	967064	607137	392863	592473	963919	628554	[illegible]	[illegible]
3	574512	967013	607500	392500	592770	963865	628905	[illegible]	[illegible]
4	574824	966961	607863	392137	593067	963811	629255	[illegible]	[illegible]
5	575136	966910	608225	391775	593363	963757	629606	[illegible]	[illegible]
6	575447	966859	608588	391412	593659	963704	629956	[illegible]	[illegible]
7	575758	966808	608950	391050	593955	963650	630306	[illegible]	[illegible]
8	576069	966756	609312	390688	594251	963596	630656	[illegible]	[illegible]
9	576379	966705	609674	390326	594547	963542	631005	[illegible]	[illegible]
10	576689	966653	610036	389964	594842	963488	631355	[illegible]	[illegible]
11	9.576999	9.966602	9.610397	10.389603	9.595137	9.963434	9.631704	[illegible]	[illegible]
12	577309	966550	610759	389241	595432	963379	632053	[illegible]	[illegible]
13	577618	966499	611120	388880	595727	963325	632401	[illegible]	[illegible]
14	577927	966447	611480	388520	596021	963271	632750	[illegible]	[illegible]
15	578236	966395	611841	388159	596315	963217	633098	[illegible]	[illegible]
16	578545	966344	612201	387799	596609	963163	633447	[illegible]	[illegible]
17	578853	966292	612561	387439	596903	963108	633795	[illegible]	[illegible]
18	579162	966240	612921	387079	597196	963054	634143	[illegible]	[illegible]
19	579470	966188	613281	386719	597490	962999	634490	[illegible]	[illegible]
20	579777	966136	613641	386359	597783	962945	634838	[illegible]	[illegible]
21	9.580085	9.966085	9.614000	10.386000	9.598075	9.962890	9.635185	[illegible]	[illegible]
22	580392	966033	614359	385641	598368	962836	635532	[illegible]	[illegible]
23	580699	965981	614718	385282	598660	962781	635879	[illegible]	[illegible]
24	581005	965929	615077	384923	598952	962727	636226	[illegible]	[illegible]
25	581312	965876	615435	384565	599244	962672	636572	[illegible]	[illegible]
26	581618	965824	615793	384207	599536	962617	636919	[illegible]	[illegible]
27	581924	965772	616151	383849	599827	962562	637265	[illegible]	[illegible]
28	582229	965720	616509	383491	600118	962508	637611	[illegible]	32
29	582535	965668	616867	383133	600409	962453	637956	[illegible]	31
30	582840	965615	617224	382776	600700	962398	638302	[illegible]	30
31	9.583145	9.965563	9.617582	10.382418	9.600990	9.962343	9.638647	10.361353	[illegible]
32	583449	965511	617939	382061	601280	962288	638992	[illegible]	[illegible]
33	583754	965458	618295	381705	601570	962233	639337	[illegible]	[illegible]
34	584058	965406	618652	381348	601860	962178	639682	[illegible]	[illegible]
35	584361	965353	619008	380992	602150	962123	640027	[illegible]	[illegible]
36	584665	965301	619364	380636	602439	962067	640371	[illegible]	[illegible]
37	584968	965248	619721	380279	602728	962012	640716	[illegible]	[illegible]
38	585272	965195	620076	379924	603017	961957	641060	358940	[illegible]
39	585574	965143	620432	379568	603305	961902	641404	358596	[illegible]
40	585877	965090	620787	379213	603594	961846	641747	358253	[illegible]
41	9.586179	9.965037	9.621142	10.378858	9.603882	9.961791	9.642091	10.357909	[illegible]
42	586482	964984	621497	378503	604170	961735	642434	357566	18
43	586783	964931	621852	378148	604457	961680	642777	357223	17
44	587085	964879	622207	377793	604745	961624	643120	356880	16
45	587386	964826	622561	377439	605032	961569	643463	356537	15
46	587688	964773	622915	377085	605319	961513	643806	356194	14
47	587989	964720	623269	376731	605606	961458	644148	355852	13
48	588289	964666	623623	376377	605892	961402	644490	355510	12
49	588590	964613	623976	376024	606179	961346	644832	355168	11
50	588890	964560	624330	375670	606465	961290	645174	354826	10
51	9.589190	9.964507	9.624683	10.375317	9.606751	9.961235	9.645516	10.354484	9
52	589489	964454	625036	374964	607036	961179	645857	354143	8
53	589789	964400	625388	374612	607322	961123	646199	353801	7
54	590088	964347	625741	374259	607607	961067	646540	353460	6
55	590387	964294	626093	373907	607892	961011	646881	353119	5
56	590686	964240	626445	373555	608177	960955	647222	352778	4
57	590984	964187	626797	373203	608461	960899	647562	352438	3
58	591282	964133	627149	372851	608745	960843	647903	352097	2
59	591580	964080	627501	372499	609029	960786	648243	351757	1
60	591878	964026	627852	372148	609313	960730	648583	351417	0
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	67 Degrees.				66 Degrees.				

	24 Degrees.			25 Degrees.				
	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
3	9.960730	9.648583	10.351417	9.625948	9.957276	9.668673	10.331327	60
7	960674	648923	351077	626219	957217	669002	330998	59
0	960618	649263	350737	626490	957158	669332	330668	58
4	960561	649602	350398	626760	957099	669661	330339	57
7	960505	649942	350058	627030	957040	669991	330009	56
9	960448	650281	349719	627300	956981	670320	329680	55
2	960392	650620	349380	627570	956921	670649	329351	54
4	960335	650959	349041	627840	956862	670977	329023	53
6	960279	651297	348703	628109	956803	671306	328694	52
8	960222	651636	348364	628378	956744	671634	328366	51
0	960165	651974	348026	628647	956684	671963	328037	50
1	9.960109	9.652312	10.347688	9.628916	9.956625	9.672291	10.327709	49
2	960052	652650	347350	629185	956566	672619	327381	48
3	959995	652988	347012	629453	956506	672947	327053	47
4	959938	653326	346674	629721	956447	673274	326726	46
5	959882	653663	346337	629989	956387	673602	326398	45
5	959825	654000	346000	630257	956327	673929	326071	44
5	959768	654337	345663	630524	956268	674257	325743	43
5	959711	654674	345326	630792	956208	674584	325416	42
5	959654	655011	344989	631059	956148	674910	325090	41
4	959596	655348	344652	631326	956089	675237	324763	40
3	9.959539	9.655684	10.344316	9.631593	9.956029	9.675564	10.324436	39
2	959482	656020	343980	631859	955969	675890	324110	38
1	959425	656356	343644	632125	955909	676217	323783	37
0	959368	656692	343308	632392	955849	676543	323457	36
8	959310	657028	342972	632658	955789	676869	323131	35
6	959253	657364	342636	632923	955729	677194	322806	34
4	959195	657699	342301	633189	955669	677520	322480	33
2	959138	658034	341966	633454	955609	677846	322154	32
0	959080	658369	341631	633719	955548	678171	321829	31
7	959023	658704	341296	633984	955488	678496	321504	30
4	9.958965	9.659039	10.340961	9.634249	9.955428	9.678821	10.321179	29
1	958908	659373	340627	634514	955368	679146	320854	28
8	958850	659708	340292	634778	955307	679471	320529	27
4	958792	660042	339958	635042	955247	679795	320205	26
0	958734	660376	339624	635306	955186	680120	319880	25
6	958677	660710	339290	635570	955126	680444	319556	24
2	958619	661043	338957	635834	955065	680768	319232	23
8	958561	661377	338623	636097	955005	681092	318908	22
3	958503	661710	338290	636360	954944	681416	318584	21
8	958445	662043	337957	636623	954883	681740	318260	20
3	9.958387	9.662376	10.337624	9.636886	9.954823	9.682063	10.317937	19
8	958329	662709	337291	637148	954762	682387	317613	18
3	958271	663042	336958	637411	954701	682710	317290	17
7	958213	663375	336625	637673	954640	683033	316967	16
1	958154	663707	336293	637935	954579	683356	316644	15
5	958096	664039	335961	638197	954518	683679	316321	14
9	958038	664371	335629	638458	954457	684001	315999	13
2	957979	664703	335297	638720	954396	684324	315676	12
6	957921	665035	334965	638981	954335	684646	315354	11
9	957863	665366	334634	639242	954274	684968	315032	10
2	9.957804	9.665697	10.334303	9.639503	9.954213	9.685290	10.314710	9
4	957746	666029	333971	639764	954152	685612	314388	8
7	957687	666360	333640	640024	954090	685934	314066	7
9	957628	666691	333309	640284	954029	686255	313745	6
1	957570	667021	332979	640544	953968	686577	313423	5
3	957511	667352	332648	640804	953906	686898	313102	4
5	957452	667682	332318	641064	953845	687219	312781	3
6	957393	668013	331987	641324	953783	687540	312460	2
7	957335	668343	331657	641583	953722	687861	312139	1
8	957276	668672	331328	641842	953660	688182	311818	0
	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
	65 Degrees.			64 Degrees.				

	26 Degrees.				27 Degrees.				
	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
	9.641842	9.953660	9.688182	10.311818	9.657047	9.949881	9.707166	10.292834	60
	642101	953599	688502	311498	657295	949816	707478	292522	59
	642360	953537	688823	311177	657542	949752	707790	292210	58
	642618	953475	689143	310857	657790	949688	708102	291898	57
	642877	953413	689463	310537	658037	949623	708414	291586	56
	643135	953352	689783	310217	658284	949558	708726	291274	55
	643393	953290	690103	309897	658531	949494	709037	290963	54
	643650	953228	690423	309577	658778	949429	709349	290651	53
	643908	953166	690742	309258	659025	949364	709660	290340	52
	644165	953104	691062	308938	659271	949300	709971	290029	51
	644423	953042	691381	308619	659517	949235	710282	289718	50
	9.644680	9.952980	9.691700	10.308300	9.659763	9.949170	9.710593	10.289407	49
	644936	952918	692019	307981	660009	949105	710904	289096	48
	645193	952855	692338	307662	660255	949040	711215	288785	47
	645450	952793	692656	307344	660501	948975	711525	288475	46
	645706	952731	692975	307025	660746	948910	711836	288164	45
	645962	952669	693293	306707	660991	948845	712146	287854	44
	646218	952606	693612	306388	661236	948780	712456	287544	43
	646474	952544	693930	306070	661481	948715	712766	287234	42
	646729	952481	694248	305752	661726	948650	713076	286924	41
	646984	952419	694566	305434	661970	948584	713386	286614	40
1	9.647240	9.952356	9.694883	10.305117	9.662214	9.948519	9.713696	10.286304	39
2	647494	952294	695201	304799	662459	948454	714005	285995	38
3	647749	952231	695518	304482	662703	948388	714314	285686	37
4	648004	952168	695836	304164	662946	948323	714624	285376	36
5	648258	952106	696153	303847	663190	948257	714933	285067	35
6	648512	952043	696470	303530	663433	948192	715242	284758	34
7	648766	951980	696787	303213	663677	948126	715551	284449	33
8	649020	951917	697103	302897	663920	948060	715860	284140	32
9	649274	951854	697420	302580	664163	947995	716168	283832	31
0	649527	951791	697736	302264	664406	947929	716477	283523	30
1	9.649781	9.951728	9.698053	10.301947	9.664648	9.947863	9.716785	10.283215	
2	650034	951665	698369	301631	664891	947797	717093	282907	
3	650287	951602	698685	301315	665133	947731	717401	282599	
4	650539	951539	699001	300999	665375	947665	717709	282291	
5	650792	951476	699316	300684	665617	947600	718017	281983	
6	651044	951412	699632	300368	665859	947533	718325	281675	
7	651297	951349	699947	300053	666100	947467	718633	281367	
8	651549	951286	700263	299737	666342	947401	718940	281060	
9	651800	951222	700578	299422	666583	947335	719248	280752	21
0	652052	951159	700893	299107	666824	947269	719555	280445	20
1	9.652304	9.951096	9.701208	10.298792	9.667065	9.947203	9.719862	10.280138	19
2	652555	951032	701523	298477	667305	947136	720169	279831	18
3	652806	950968	701837	298163	667546	947070	720476	279524	17
4	653057	950905	702152	297848	667786	947004	720783	279217	16
5	653308	950841	702466	297534	668027	946937	721089	278911	15
6	653558	950778	702780	297220	668267	946871	721396	278604	14
7	653808	950714	703095	296905	668506	946804	721702	278298	13
8	654059	950650	703409	296591	668746	946738	722009	277991	12
9	654309	950586	703723	296277	668986	946671	722315	277685	11
0	654558	950522	704036	295964	669225	946604	722621	277379	10
1	9.654808	9.950458	9.704350	10.295650	9.669464	9.946538	9.722927	10.277073	9
2	655058	950394	704663	295337	669703	946471	723232	276768	8
3	655307	950330	704977	295023	669942	946404	723538	276462	7
4	655556	950266	705290	294710	670181	946337	723844	276156	6
5	655805	950202	705603	294397	670419	946270	724149	275851	5
6	656054	950138	705916	294084	670658	946203	724454	275546	4
7	656302	950074	706228	293772	670896	946136	724759	275241	3
8	656551	950010	706541	293459	671134	946069	725065	274935	2
9	656799	949945	706854	293146	671372	946002	725369	274631	1
0	657047	949881	707166	292834	671609	945935	725674	274326	0
	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	
	63 Degrees.				62 Degrees.				

28 Degrees.			29 Degrees.				
osine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
45935	9.725674	10.274326	9.685571	9.941819	9.743752	10.256248	60
45868	725979	274021	685799	941749	744050	255950	59
45800	726284	273716	686027	941679	744348	255652	58
45733	726588	273412	686254	941609	744645	255355	57
45666	726892	273108	686482	941539	744943	255057	56
45598	727197	272803	686709	941469	745240	254760	55
45531	727501	272499	686936	941398	745538	254462	54
45464	727805	272195	687163	941328	745835	254165	53
45396	728109	271891	687389	941258	746132	253868	52
45328	728412	271588	687616	941187	746429	253571	51
45261	728716	271284	687843	941117	746726	253274	50
45193	9.729020	10.270980	9.688069	9.941046	9.747023	10.252977	49
45125	729323	270677	688295	940975	747319	252681	48
45058	729626	270374	688521	940905	747616	252384	47
44990	729929	270071	688747	940834	747913	252087	46
44922	730233	269767	688972	940763	748209	251791	45
44854	730535	269465	689198	940693	748505	251495	44
44786	730838	269162	689423	940622	748801	251199	43
44718	731141	268859	689648	940551	749097	250903	42
44650	731444	268556	689873	940480	749393	250607	41
44582	731746	268254	690098	940409	749689	250311	40
44514	9.732048	10.267952	9.690323	9.940338	9.749985	10.250015	39
44446	732351	267649	690548	940267	750281	249719	38
44377	732653	267347	690772	940196	750576	249424	37
44309	732955	267045	690996	940125	750872	249128	36
44241	733257	266743	691220	940054	751167	248833	35
44172	733558	266442	691444	939982	751462	248538	34
44104	733860	266140	691668	939911	751757	248243	33
44036	734162	265838	691892	939840	752052	247948	32
43967	734463	265537	692115	939768	752347	247653	31
43899	734764	265236	692339	939697	752642	247358	30
43830	9.735066	10.264934	9.692562	9.939625	9.752937	10.247063	29
43761	735367	264633	692785	939554	753231	246769	28
43693	735668	264332	693008	939482	753526	246474	27
43624	735969	264031	693231	939410	753820	246180	26
43555	736269	263731	693453	939339	754115	245885	25
43486	736570	263430	693676	939267	754409	245591	24
43417	736871	263129	693898	939195	754703	245297	23
43348	737171	262829	694120	939123	754997	245003	22
43279	737471	262529	694342	939052	755291	244709	21
43210	737771	262229	694564	938980	755585	244415	20
43141	9.738071	10.261929	9.694786	9.938908	9.755878	10.244122	19
43072	738371	261629	695007	938836	756172	243828	18
43003	738671	261329	695229	938763	756465	243535	17
42934	738971	261029	695450	938691	756759	243241	16
42864	739271	260729	695671	938619	757052	242948	15
42795	739570	260430	695892	938547	757345	242655	14
42726	739870	260130	696113	938475	757638	242362	13
42656	740169	259831	696334	938402	757931	242069	12
42587	740468	259532	696554	938330	758224	241776	11
42517	740767	259233	696775	938258	758517	241483	10
42448	9.741066	10.258934	9.696995	9.938185	9.758810	10.241190	9
42378	741365	258635	697215	938113	759102	240898	8
42308	741664	258336	697435	938040	759395	240605	7
42239	741962	258038	697654	937967	759687	240313	6
42169	742261	257739	697874	937895	759979	240021	5
42099	742559	257441	698094	937822	760272	239728	4
42029	742858	257142	698313	937749	760564	239436	3
41959	743156	256844	698532	937676	760856	239144	2
41889	743454	256546	698751	937604	761148	238852	1
41819	743752	256248	698970	937531	761439	238561	0
ne.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
Degrees.			60 Degrees.				

30 Degrees.				31 Degrees.			
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
9.698970	9.937531	9.761439	10.238561	9.711839	9.933066	9.778774	[illegible]
699189	937458	761731	238269	712050	932990	779060	[illegible]
699407	937385	762023	237977	712260	932914	779346	[illegible]
699626	937312	762314	237686	712469	932838	779632	[illegible]
699844	937238	762606	237394	712679	932762	779918	[illegible]
700062	937165	762897	237103	712889	932685	780203	[illegible]
700280	937092	763188	236812	713098	932609	780489	[illegible]
700498	937019	763479	236521	713308	932533	780775	[illegible]
700716	936946	763770	236230	713517	932457	781060	[illegible]
700933	936872	764061	235939	713726	932380	781346	[illegible]
701151	936799	764352	235648	713935	932304	781631	[illegible]
9.701368	9.936725	9.764643	10.235357	9.714144	9.932228	9.781916	[illegible]
701585	936652	764933	235067	714352	932151	782201	[illegible]
701802	936578	765224	234776	714561	932075	782486	[illegible]
702019	936505	765514	234486	714769	931998	782771	[illegible]
702236	936431	765805	234195	714978	931921	783056	216944
702452	936357	766095	233905	715186	931845	783341	216659
702669	936284	766385	233615	715394	931768	783626	216374
702885	936210	766675	233325	715602	931691	783910	216090
703101	936136	766965	233035	715809	931614	784195	215805
703317	936062	767255	232745	716017	931537	784479	215521
9.703533	9.935988	9.767545	10.232455	9.716224	9.931460	9.784764	10.215236
703749	935914	767834	232166	716432	931383	785048	214952
703964	935840	768124	231876	716639	931306	785332	214668
704179	935766	768414	231586	716846	931229	785616	214384
704395	935692	768703	231297	717053	931152	785900	214100
704610	935618	768992	231008	717259	931075	786184	213816
704825	935543	769281	230719	717466	930998	786468	213532
705040	935469	769570	230430	717673	930921	786752	213248
705254	935395	769860	230140	717879	930843	787036	[illegible]
705469	935320	770148	229852	718085	930766	787319	[illegible]
9.705683	9.935246	9.770437	10.229563	9.718291	9.930688	9.787603	[illegible]
705898	935171	770726	229274	718497	930611	787886	[illegible]
706112	935097	771015	228985	718703	930533	788170	[illegible]
706326	935022	771303	228697	718909	930456	788453	[illegible]
706539	934948	771592	228408	719114	930378	788736	211264
706753	934873	771880	228120	719320	930300	789019	210981
706967	934798	772168	227832	719525	930223	789302	210698
707180	934723	772457	227543	719730	930145	789585	210415
707393	934649	772745	227255	719935	930067	789868	210132
707606	934574	773033	226967	720140	929989	790151	209849
9.707819	9.934499	9.773321	10.226679	9.720345	9.929911	9.790433	10.209567
708032	934424	773608	226392	720549	929833	790716	209284
708245	934349	773896	226104	720754	929755	790999	209001
708458	934274	774184	225816	720958	929677	791281	208719
708670	934199	774471	225529	721162	929599	791563	208437
708882	934123	774759	225241	721366	929521	791846	208154
709094	934048	775046	224954	721570	929442	792128	207872
709306	933973	775333	224667	721774	929364	792410	207590
709518	933898	775621	224379	721978	929286	792692	207308
709730	933822	775908	224092	722181	929207	792974	207026
9.709941	9.933747	9.776195	10.223805	9.722385	9.929129	9.793256	10.206744
710153	933671	776482	223518	722588	929050	793538	206462
710364	933596	776769	223231	722791	928972	793819	206181
710575	933520	777055	222945	722994	928893	794101	205899
710786	933445	777342	222658	723197	928815	794383	205617
710997	933369	777628	222372	723400	928736	794664	205336
711208	933293	777915	222085	723603	928657	794945	205055
711419	933217	778201	221799	723805	928578	795227	204773
711629	933141	778487	221513	724007	928499	795508	204492
711839	933066	778774	221226	724210	928420	795789	204211
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.
59 Degrees.				58 Degrees.			

	32 Degrees.			33 Degrees.				
	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
)	9.928420	9.795789	10.204211	9.736109	9.923591	9.812517	10.187483	60
2	928342	796070	203930	736303	923509	812794	187206	59
4	928263	796351	203649	736498	923427	813070	186930	58
6	928183	796632	203368	736692	923345	813347	186653	57
7	928104	796913	203087	736886	923263	813623	186377	56
9	928025	797194	202806	737080	923181	813899	186101	55
0	927946	797475	202525	737274	923098	814175	185825	54
2	927867	797755	202245	737467	923016	814452	185548	53
3	927787	798036	201964	737661	922933	814728	185272	52
4	927708	798316	201684	737855	922851	815004	184996	51
5	927629	798596	201404	738048	922768	815279	184721	50
6	9.927549	9.798877	10.201123	9.738241	9.922686	9.815555	10.184445	49
6	927470	799157	200843	738434	922603	815831	184169	48
7	927390	799437	200563	738627	922520	816107	183893	47
7	927310	799717	200283	738820	922438	816382	183618	46
8	927231	799997	200003	739013	922355	816658	183342	45
8	927151	800277	199723	739206	922272	816933	183067	44
8	927071	800557	199443	739398	922189	817209	182791	43
8	926991	800836	199164	739590	922106	817484	182516	42
7	926911	801116	198884	739783	922023	817759	182241	41
7	926831	801396	198604	739975	921940	818035	181965	40
7	9.926751	9.801675	10.198325	9.740167	9.921857	9.818310	10.181690	39
6	926671	801955	198045	740359	921774	818585	181415	38
5	926591	802234	197766	740550	921691	818860	181140	37
4	926511	802513	197487	740742	921607	819135	180865	36
3	926431	802792	197208	740934	921524	819410	180590	35
2	926351	803072	196928	741125	921441	819684	180316	34
1	926270	803351	196649	741316	921357	819959	180041	33
0	926190	803630	196370	741508	921274	820234	179766	32
8	926110	803908	196092	741699	921190	820508	179492	31
7	926029	804187	195813	741889	921107	820783	179217	30
5	9.925949	9.804466	10.195534	9.742080	9.921023	9.821057	10.178943	29
3	925868	804745	195255	742271	920939	821332	178668	28
1	925788	805023	194977	742462	920856	821606	178394	27
9	925707	805302	194698	742652	920772	821880	178120	26
6	925626	805580	194420	742842	920688	822154	177846	25
4	925545	805859	194141	743033	920604	822429	177571	24
2	925465	806137	193863	743223	920520	822703	177297	23
9	925384	806415	193585	743413	920436	822977	177023	22
6	925303	806693	193307	743602	920352	823250	176750	21
3	925222	806971	193029	743792	920268	823524	176476	20
00	9.925141	9.807249	10.192751	9.743982	9.920184	9.823798	10.176202	19
7	925060	807527	192473	744171	920099	824072	175928	18
4	924979	807805	192195	744361	920015	824345	175655	17
30	924897	808083	191917	744550	919931	824619	175381	16
77	924816	808361	191639	744739	919846	824893	175107	15
73	924735	808638	191362	744928	919762	825166	174834	14
69	924654	808916	191084	745117	919677	825439	174561	13
65	924572	809193	190807	745306	919593	825713	174287	12
61	924491	809471	190529	745494	919508	825986	174014	11
57	924409	809748	190252	745683	919424	826259	173741	10
53	9.924328	9.810025	10.189975	9.745871	9.919339	9.826532	10.173468	9
49	924246	810302	189698	746060	919254	826805	173195	8
44	924164	810580	189420	746248	919169	827078	172922	7
39	924083	810857	189143	746436	919085	827351	172649	6
35	924001	811134	188866	746624	919000	827624	172376	5
30	923919	811410	188590	746812	918915	827897	172103	4
25	923837	811687	188313	746999	918830	828170	171830	3
19	923755	811964	188036	747187	918745	828442	171558	2
14	923673	812241	187759	747374	918659	828715	[illegible]	[illegible]
09	923591	812517	187483	747562	918574	828987	[illegible]	[illegible]
e.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Ta	
	57 Degrees.				56 Degrees.			

	34 Degrees.				35 Degrees.		
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.
0	9.747562	9.918574	9.828987	10.171013	9.758591	9.913365	9.84522
1	747749	918489	829260	170740	758772	913276	84549
2	747936	918404	829532	170468	758952	913187	84576
3	748123	918318	829805	170195	759132	913099	84603
4	748310	918233	830077	169923	759312	913010	84630
5	748497	918147	830349	169651	759492	912922	846570
6	748683	918062	830621	169379	759672	912833	84683
7	748870	917976	830893	169107	759852	912744	847107
8	749056	917891	831165	168835	760031	912655	84737
9	749243	917805	831437	168563	760211	912566	847644
10	749429	917719	831709	168291	760390	912477	847913
11	9.749615	9.917634	9.831981	10.168019	9.760569	9.912388	9.848181
12	749801	917548	832253	167747	760748	912299	848449
13	749987	917462	832525	167475	760927	912210	848717
14	750172	917376	832796	167204	761106	912121	84898
15	750358	917290	833068	166932	761285	912031	849254
16	750543	917204	833339	166661	761464	911942	849522
17	750729	917118	833611	166389	761642	911853	849790
18	750914	917032	833882	166118	761821	911763	850058
19	751099	916946	834154	165846	761999	911674	850325
20	751284	916859	834425	165575	762177	911584	850593
21	9.751469	9.916773	9.834696	10.165304	9.762356	9.911495	9.850861
22	751654	916687	834967	165033	762534	911405	851129
23	751839	916600	835238	164762	762712	911315	851396
24	752023	916514	835509	164491	762889	911226	851664
25	752208	916427	835780	164220	763067	911136	851931
26	752392	916341	836051	163949	763245	911046	852199
27	752576	916254	836322	163678	763422	910956	852466
28	752760	916167	836593	163407	763600	910866	852733
29	752944	916081	836864	163136	763777	910776	853001
30	753128	915994	837134	162866	763954	910686	853268
31	9.753312	9.915907	9.837405	10.162595	9.764131	9.910596	9.853535
32	753495	915820	837675	162325	764308	910506	853802
33	753679	915733	837946	162054	764485	910415	854069
34	753862	915646	838216	161784	764662	910325	854336
35	754046	915559	838487	161513	764838	910235	854603
36	754229	915472	838757	161243	765015	910144	854870
37	754412	915385	839027	160973	765191	910054	855137
38	754595	915297	839297	160703	765367	909963	855404
39	754778	915210	839568	160432	765544	909873	855671
40	754960	915123	839838	160162	765720	909782	855938
41	9.755143	9.915035	9.840108	10.159892	9.765896	9.909691	9.856204
42	755326	914948	840378	159622	766072	909601	856471
43	755508	914860	840647	159353	766247	909510	856737
44	755690	914773	840917	159083	766423	909419	857004
45	755872	914685	841187	158813	766598	909328	857270
46	756054	914598	841457	158543	766774	909237	857537
47	756236	914510	841726	158274	766949	909146	857803
48	756418	914422	841996	158004	767124	909055	858069
49	756600	914334	842266	157734	767300	908964	858336
50	756782	914246	842535	157465	767475	908873	858602
51	9.756963	9.914158	9.842805	10.157195	9.767649	9.908781	9.858868
52	757144	914070	843074	156926	767824	908690	859134
53	757326	913982	843343	156657	767999	908599	859400
54	757507	913894	843612	156388	768173	908507	859666
55	757688	913806	843882	156118	768348	908416	859932
56	757869	913718	844151	155849	768522	908324	860198
57	758050	913630	844420	155580	768697	908233	860464
58	758230	913541	844689	155311	768871	908141	860730
59	758411	913453	844958	155042	769045	908049	860995
60	758591	913365	845227	154773	769219	907958	8612
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cota
	55 Degrees.				54 Deg		

36 Degrees.			37 Degrees.				
Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	′
9.907958	9.861261	10.138739	9.779463	9.902349	9.877114	10.122886	60
907866	861527	138473	779631	902253	877377	122623	59
907774	861792	138208	779798	902158	877640	122360	58
907682	862058	137942	779966	902063	877903	122097	57
907590	862323	137677	780133	901967	878165	121835	56
907498	862589	137411	780300	901872	878428	121572	55
907406	862854	137146	780467	901776	878691	121309	54
907314	863119	136881	780634	901681	878953	121047	53
907222	863385	136615	780801	901585	879216	120784	52
907129	863650	136350	780968	901490	879478	120522	51
907037	863915	136085	781134	901394	879741	120259	50
9.906945	9.864180	10.135820	9.781301	9.901298	9.880003	10.119997	49
906852	864445	135555	781468	901202	880265	119735	48
906760	864710	135290	781634	901106	880528	119472	47
906667	864975	135025	781800	901010	880790	119210	46
906575	865240	134760	781966	900914	881052	118948	45
906482	865505	134495	782132	900818	881314	118686	44
906389	865770	134230	782298	900722	881576	118424	43
906296	866035	133965	782464	900626	881839	118161	42
906204	866300	133700	782630	900529	882101	117899	41
906111	866564	133436	782796	900433	882363	117637	40
9.906018	9.866829	10.133171	9.782961	9.900337	9.882625	10.117375	39
905925	867094	132906	783127	900240	882887	117113	38
905832	867358	132642	783292	900144	883148	116852	37
905739	867623	132377	783458	900047	883410	116590	36
905645	867887	132113	783623	899951	883672	116328	35
905552	868152	131848	783788	899854	883934	116066	34
905459	868416	131584	783953	899757	884196	115804	33
905366	868680	131320	784118	899660	884457	115543	32
905272	868945	131055	784282	899564	884719	115281	31
905179	869209	130791	784447	899467	884980	115020	30
9.905085	9.869473	10.130527	9.784612	9.899370	9.885242	10.114758	29
904992	869737	130263	784776	899273	885503	114497	28
904898	870001	129999	784941	899176	885765	114235	27
904804	870265	129735	785105	899078	886026	113974	26
904711	870529	129471	785269	898981	886288	113712	25
904617	870793	129207	785433	898884	886549	113451	24
904523	871057	128943	785597	898787	886810	113190	23
904429	871321	128679	785761	898689	887072	112928	22
904335	871585	128415	785925	898592	887333	112667	21
904241	871849	128151	786089	898494	887594	112406	20
9.904147	9.872112	10.127888	9.786252	9.898397	9.887855	10.112145	19
904053	872376	127624	786416	898299	888116	111884	18
903959	872640	127360	786579	898202	888377	111623	17
903864	872903	127097	786742	898104	888639	111361	16
903770	873167	126833	786906	898006	888900	111100	15
903676	873430	126570	787069	897908	889160	110840	14
903581	873694	126306	787232	897810	889421	110579	13
903487	873957	126043	787395	897712	889682	110318	12
903392	874220	125780	787557	897614	889943	110057	11
903298	874484	125516	787720	897516	890204	109796	10
9.903203	9.874747	10.125253	9.787883	9.897418	9.890465	10.109535	9
903108	875010	124990	788045	897320	890725	109275	8
903014	875273	124727	788208	897222	890986	109014	7
902919	875536	124464	788370	897123	891247	108753	6
902824	875800	124200	788532	897025	891507	108493	5
902729	876063	123937	788694	896926	891768	108232	4
902634	876326	123674	788856	896828	892028	107972	3
902539	876589	123411	789018	896729	892289	107711	2
902444	876851	123149	789180	896631	892549	107451	1
902349	877114	122886	789342	896532	892810	107190	0
Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	′
53 Degrees.			52 Degrees.				

	38 Degrees.				39 Degrees.			
′	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
0	9.789342	9.896532	9.892810	10.107190	9.798872	9.890503	9.90836	10.0916
1	789504	896433	893070	106930	799028	890400	90362	0913
2	789665	896335	893331	106669	799184	890298	90888	0911
3	789827	896236	893591	106409	799339	890195	90914	0908
4	789988	896137	893851	106149	799495	890093	90940	0906
5	790149	896038	894111	105889	799651	889990	90966	[illegible]
6	790310	895939	894371	105629	799806	889888	90991	[illegible]
7	790471	895840	894632	105368	799962	889785	910177	[illegible]
8	790632	895741	894892	105108	800117	889682	910435	[illegible]
9	790793	895641	895152	104848	800272	889579	910693	[illegible]
10	790954	895542	895412	104588	800427	889477	910951	[illegible]
11	9.791115	9.895443	9.895672	10.104328	9.800582	9.889374	9.911209	[illegible]
12	791275	895343	895932	104068	800737	889271	911467	[illegible]
13	791436	895244	896192	103808	800892	889168	911724	[illegible]
14	791596	895145	896452	103548	801047	889064	911982	[illegible]
15	791757	895045	896712	103288	801201	888961	912240	[illegible]
16	791917	894945	896971	103029	801356	888858	912498	[illegible]
17	792077	894846	897231	102769	801511	888755	912756	[illegible]
18	792237	894746	897491	102509	801665	888651	913014	[illegible]
19	792397	894646	897751	102249	801819	888548	913271	[illegible]
20	792557	894546	898010	101990	801973	888444	913529	[illegible]
21	9.792716	9.894446	9.898270	10.101730	9.802128	9.888341	9.913787	[illegible]
22	792876	894346	898530	101470	802282	888237	914044	[illegible]
23	793035	894246	898789	101211	802436	888134	914302	[illegible]
24	793195	894146	899049	100951	802589	888030	914560	[illegible]
25	793354	894046	899308	100692	802743	887926	914817	[illegible]
26	793514	893946	899568	100432	802897	887822	915075	[illegible]
27	793673	893846	899827	100173	803050	887718	915332	[illegible]
28	793832	893745	900086	099914	803204	887614	915590	[illegible]
29	793991	893645	900346	099654	803357	887510	915847	[illegible]
30	794150	893544	900605	099395	803511	887406	916104	[illegible]
31	9.794308	9.893444	9.900864	10.099136	9.803664	9.887302	9.916362	[illegible]
32	794467	893343	901124	098876	803817	887198	916619	[illegible]
33	794626	893243	901383	098617	803970	887093	91687	[illegible]
34	794784	893142	901642	098358	804123	886989	91713	[illegible]
35	794942	893041	901901	098099	804276	886885	9173	[illegible]
36	795101	892940	902160	097840	804428	886780	9176	[illegible]
37	795259	892839	902419	097581	804581	886676	917	[illegible]
38	795417	892739	902679	097321	804734	886571	91	[illegible]
39	795575	892638	902938	097062	804886	886466	91	[illegible]
40	795733	892536	903197	096803	805039	886362	91	[illegible]
41	9.795891	9.892435	9.903455	10.096545	9.805191	9.886257	9.9	[illegible]
42	796049	892334	903714	096286	805343	886152	[illegible]	[illegible]
43	796206	892233	903973	096027	805495	886047	[illegible]	[illegible]
44	796364	892132	904232	095768	805647	885942	[illegible]	[illegible]
45	796521	892030	904491	095509	805799	885837	[illegible]	[illegible]
46	796679	891929	904750	095250	805951	885732	[illegible]	[illegible]
47	296836	891827	905008	094992	806103	885627	[illegible]	[illegible]
48	796993	891726	905267	094733	806254	885522	[illegible]	[illegible]
49	797150	891624	905526	094474	806406	885416	[illegible]	[illegible]
50	797307	891523	905784	094216	806557	88531	[illegible]	[illegible]
51	9.797464	9.891421	9.906043	10.093957	9.806709	9.88520	[illegible]	[illegible]
52	797621	891319	906302	093698	806860	8851	[illegible]	[illegible]
53	797777	891217	906560	093440	807011	8849	[illegible]	[illegible]
54	797934	891115	906819	093181	807163	8848	[illegible]	[illegible]
55	798091	891013	907077	092923	807314	884	[illegible]	[illegible]
56	798247	890911	907336	092664	807465	88	[illegible]	[illegible]
57	798403	890809	907594	092406	807615	88	[illegible]	[illegible]
58	798560	890707	907852	092148	807766	[illegible]	[illegible]	[illegible]
59	798716	890605	908111	091889	807917	[illegible]	[illegible]	[illegible]
60	798872	890503	908369	091631	808067	[illegible]	[illegible]	[illegible]
′	Cosine.	Sine.	Cotang.	Tang.	Cosine.			

51 Degrees.

egrees.			41 Degrees.				
.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
54	9.923813	10.076187	9.816943	9.877780	9.939163	10.060837	60
48	924070	075930	817088	877670	939418	060582	59
42	924327	075673	817233	877560	939673	060327	58
36	924583	075417	817379	877450	939928	060072	57
29	924840	075160	817524	877340	940183	059817	56
23	925096	074904	817668	877230	940438	059562	55
17	925352	074648	817813	877120	940694	059306	54
10	925609	074391	817958	877010	940949	059051	53
04	925865	074135	818103	876899	941204	058796	52
97	926122	073878	818247	876789	941458	058542	51
91	926378	073622	818392	876678	941714	058286	50
84	9.926634	10.073366	9.818536	9.876568	9.941968	10.058032	49
77	926890	073110	818681	876457	942223	057777	48
71	927147	072853	818825	876347	942478	057522	47
64	927403	072597	818969	876236	942733	057267	46
57	927659	072341	819113	876125	942988	057012	45
50	927915	072085	819257	876014	943243	056757	44
43	928171	071829	819401	875904	943498	056502	43
36	928427	071573	819545	875793	943752	056248	42
29	928683	071317	819689	875682	944007	055993	41
21	928940	071060	819832	875571	944262	055738	40
14	9.929196	10.070804	9.819976	9.875459	9.944517	10.055483	39
07	929452	070548	820120	875348	944771	055229	38
99	929708	070292	820263	875237	945026	054974	37
92	929964	070036	820406	875126	945281	054719	36
84	930220	069780	820550	875014	945535	054465	35
77	930475	069525	820693	874903	945790	054210	34
69	930731	069269	820836	874791	946045	053955	33
61	930987	069013	820979	874680	946299	053701	32
53	931243	068757	821122	874568	946554	053446	31
46	931499	068501	821265	874456	946808	053192	30
38	9.931755	10.068245	9.821407	9.874344	9.947063	10.052937	29
30	932010	067990	821550	874232	947318	052682	28
22	932266	067734	821693	874121	947572	052428	27
13	932522	067478	821835	874009	947826	052174	26
05	932778	067222	821977	873896	948081	051919	25
97	933033	066967	822120	873784	948336	051664	24
89	933289	066711	822262	873672	948590	051410	23
80	933545	066455	822404	873560	948844	051156	22
72	933800	066200	822546	873448	949099	050901	21
63	934056	065944	822688	873335	949353	050647	20
55	9.934311	10.065689	9.822830	9.873223	9.949607	10.050393	19
46	934567	065433	822972	873110	949862	050138	18
37	934823	065177	823114	872998	950116	049884	17
29	935078	064922	823255	872885	950370	049630	16
20	935333	064667	823397	872772	950625	049375	15
11	935589	064411	823539	872659	950879	049121	14
02	935844	064156	823680	872547	951133	048867	13
93	936100	063900	823821	872434	951388	048612	12
84	936355	063645	823963	872321	951642	048358	11
75	936610	063390	824104	872208	951896	048104	10
66	9.936866	10.063134	9.824245	9.872095	9.952150	10.047850	9
56	937121	062879	824386	871981	952405	047595	8
47	937376	062624	824527	871868	952659	047341	7
38	937632	062368	824668	871755	952913	047087	6
28	937887	062113	824808	871641	953167	046833	5
19	938142	061858	824949	871528	953421	046579	4
09	938398	061602	825090	871414	953675	046325	3
99	938653	061347	825230	871301	953929	046071	2
90	938908	061092	825371	871187	954183	045817	1
80	939163	060837	825511	871073	954437	045563	0
	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.	
grees.			48 Degrees.				

42 Degrees.				43 Degrees.			
Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.
.825511	9.871073	9.954437	10.045563	9.833783	9.864127	9.969656	[illegible]
825651	870960	954691	045309	833919	864010	969909	[illegible]
825791	870846	954945	045055	834054	863892	970162	[illegible]
825931	870732	955200	044800	834189	863774	970416	[illegible]
826071	870618	955454	044546	834325	863656	970669	[illegible]
826211	870504	955707	044293	834460	863538	970922	[illegible]
826351	870390	955961	044039	834595	863419	971175	[illegible]
826491	870276	956215	043785	834730	863301	971429	[illegible]
826631	870161	956469	043531	834865	863183	971682	[illegible]
826770	870047	956723	043277	834999	863064	971935	[illegible]
826910	869933	956977	043023	835134	862946	972188	[illegible]
.827049	9.869818	9.957231	10.042769	9.835269	9.862827	9.972441	[illegible]
827189	869704	957485	042515	835403	862709	972694	[illegible]
827328	869589	957739	042261	835538	862590	972948	[illegible]
827467	869474	957993	042007	835672	862471	973201	[illegible]
827606	869360	958246	041754	835807	862353	973454	[illegible]
827745	869245	958500	041500	835941	862234	973707	[illegible]
827884	869130	958754	041246	836075	862115	973960	[illegible]
828023	869015	959008	040992	836209	861996	974213	[illegible]
828162	868900	959262	040738	836343	861877	974466	[illegible]
828301	868785	959516	040484	836477	861758	974719	[illegible]
.828439	9.868670	9.959769	10.040231	9.836611	9.861638	9.974973	[illegible]
828578	868555	960023	039977	836745	861519	975226	[illegible]
828716	868440	960277	039723	836878	861400	975479	[illegible]
828855	868324	960531	039469	837012	861280	975732	[illegible]
828993	868209	960784	039216	837146	861161	975985	[illegible]
829131	868093	961038	038962	837279	861041	976238	[illegible]
829269	867978	961291	038709	837412	860922	976491	023509
829407	867862	961545	038455	837546	860802	976744	023256
829545	867747	961799	038201	837679	860682	976997	023003
829683	867631	962052	037948	837812	860562	977250	022750
.829821	9.867515	9.962306	10.037694	9.837945	9.860442	9.977503	10.022497
829959	867399	962560	037440	838078	860322	977756	022244
830097	867283	962813	037187	838211	860202	978009	021991
830234	867167	963067	036933	838344	860082	978262	021738
830372	867051	963320	036680	838477	859962	978515	021485
830509	866935	963574	036426	838610	859842	978768	021232
830646	866819	963827	036173	838742	859721	979021	020979
830784	866703	964081	035919	838875	859601	979274	020726
830921	866586	964335	035665	839007	859480	979527	020473
831058	866470	964588	035412	839140	859360	979780	020220
.831195	9.866353	9.964842	10.035158	9.839272	9.859239	9.980033	10.019967
831332	866237	965095	034905	839404	859119	980286	019714
831469	866120	965349	034651	839536	858998	980538	019462
831606	866004	965602	034398	839668	858877	980791	019209
831742	865887	965855	034145	839800	858756	981044	018956
831879	865770	966109	033891	839932	858635	981297	018703
832015	865653	966362	033638	840064	858514	981550	018450
832152	865536	966616	033384	840196	858393	981803	018197
832288	865419	966869	033131	840328	858272	982056	017944
832425	865302	967123	032877	840459	858151	982309	017691
.832561	9.865185	9.967376	10.032624	9.840591	9.858029	9.982562	10.017438
832697	865068	967629	032371	840722	857908	982814	017186
832833	864950	967883	032117	840854	857786	983067	016933
832969	864833	968136	031864	840985	857665	983320	016680
833105	864716	968389	031611	841116	857543	983573	[illegible]
833241	864598	968643	031357	841247	857422	983826	[illegible]
833377	864481	968896	031104	841378	857300	984079	[illegible]
833512	864363	969149	030851	841509	857178	984331	015669
833648	864245	969403	030597	841640	857056	984584	015416
833783	864127	969656	030344	841771	856934	984837	015163
Cosine.	Sine.	Cotang.	Tang.	Cosine.	Sine.	Cotang.	Tang.
47 Degrees.				46 Degrees.			

44 Degrees.

	Cosine.	Tang.	Cotang.	′
1	9.856934	9.984837	10.015163	60
2	856812	985090	014910	59
3	856690	985343	014657	58
3	856568	985596	014404	57
4	856446	985848	014152	56
4	856323	986101	013899	55
5	856201	986354	013646	54
5	856078	986607	013393	53
5	855956	986860	013140	52
6	855833	987112	012888	51
6	855711	987365	012635	50
6	9.855588	9.987618	10.012382	49
6	855465	987871	012129	48
6	855342	988123	011877	47
5	855219	988376	011624	46
5	855096	988629	011371	45
5	854973	988882	011118	44
4	854850	989134	010866	43
4	854727	989387	010613	42
3	854603	989640	010360	41
72	854480	989893	010107	40
02	9.854356	9.990145	10.009855	39
31	854233	990398	009602	38
60	864109	990651	009349	37
89	853986	990903	009097	36
18	853862	991156	008844	35
47	853738	991409	008591	34
76	853614	991662	008338	33
05	853490	991914	008086	32
33	853366	992167	007833	31
62	853242	992420	007580	30
90	9.853118	9.992672	10.007328	29
19	852994	992925	007075	28
47	852869	993178	006822	27
75	852745	993430	006570	26
04	852620	993683	006317	25
32	852496	993936	006064	24
60	852371	994189	005811	23
88	852247	994441	005559	22
16	852122	994694	005306	21
44	851997	994947	005053	20
71	9.851872	9.995199	10.004801	19
99	851747	995452	004548	18
27	851622	995705	004295	17
54	851497	995957	004043	16
82	851372	996210	003790	15
09	851246	996463	003537	14
36	851121	996715	003285	13
64	850996	996968	003032	12
91	850870	997221	002779	11
18	850745	997473	002527	10
45	9.850619	9.997726	10.002274	9
72	850493	997979	002021	8
99	850368	998231	001769	7
26	850242	998484	001516	6
52	850116	998737	001263	5
79	849990	998989	001011	4
06	849864	999242	000758	3
32	849738	999495	000505	2
59	849611	999747	000253	1
85	849485	10.000000	000000	0
	Sine.	Cotang.	Tang.	′

45 Degrees.

RULES FOR FINDING LOGARITHMIC SECANTS, VERSED SINES, &c.

I. To find the Secant.—Subtract the Log. Cosine from 20.

II. To find the Cosecant.—Subtract the Log. Sine from 20.

III. To find the Versed Sine.—Add 0.301030 to twice the Log. Sine of half the arc, and diminish the index of the sum by 10.

IV. To find the Coversed Sine.—Add 0.301030 to twice the Log. Sine of half the complement of the arc, and diminish the index of the sum by 10.

RULES FOR FINDING NATURAL SECANTS, VERSED SINES, &c.

I. To find the Secant.—Divide 1 by the Natural Cosine.

II. To find the Cosecant.—Divide 1 by the Natural Sine.

III. To find the Versed Sine.—Subtract the Natural Cosine from 1.

IV. To find the Coversed Sine.—Subtract the Natural Sine from 1.

NOTE. In France the circumference of the circle has lately been divided into 400 degrees, the degree into 100 minutes, and the minute into 100 seconds, &c. which is called the centesimal division, and is to the sexagesimal in the ratio of 9 to 10; hence, to reduce centesimal into sexagesimal degrees, &c. subtract one-tenth; and to reduce sexagesimal into centesimal degrees, add one-ninth of the arc to itself.

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	
0	000000	017452	034899	052336	069756	087156	104528	121869	139173	156434	60
5	1454	8907	6353	3788	071207	8605	5975	3313	140613	7871	55
10	2909	020361	7806	5241	2658	090053	7421	4756	2053	9307	50
15	4363	1815	9260	6693	4108	1502	8867	6199	3493	160743	45
20	5818	3269	040713	8145	5559	2950	110313	7642	4932	2178	40
25	7272	4723	2166	9597	7009	4398	1758	9084	6371	3613	35
30	8727	6177	3619	061049	8459	5846	3203	130526	7809	5048	30
35	010181	7631	5072	2500	9909	7293	4648	1968	9248	6482	25
40	1635	9085	6525	3952	081359	8741	6093	3410	150686	7916	20
45	3090	030539	7978	5403	2808	100188	7537	4851	2123	9350	15
50	4544	1992	9431	6854	4258	1635	8982	6292	3561	170783	10
55	5998	3446	050883	8306	5707	3082	120426	7733	4998	2216	5
60	7452	4899	2336	9756	7156	4528	1869	9173	6434	3648	0
Cos.	89°	88°	87°	86°	85°	84°	83°	82°	81°	80°	
Sin.	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	
0	173648	190809	207912	224951	241922	258819	275637	292372	309017	325568	60
5	5080	2237	9334	6368	3333	260224	7035	3762	310400	6943	55
10	6512	3664	210756	7784	4743	1628	8432	5152	1782	8317	50
15	7944	5090	2178	9200	6153	3031	9829	6542	3164	9691	45
20	9375	6517	3599	230616	7563	4434	281225	7930	4545	331063	40
25	180805	7942	5019	2031	8972	5837	2620	9318	5925	2435	35
30	2236	9368	6440	3445	250380	7238	4015	300706	7305	3807	30
35	3665	200793	7859	4859	1788	8640	5410	2093	8684	5178	25
40	5095	2218	9279	6273	3195	270040	6803	3479	320062	6547	20
45	6524	3642	220697	7686	4602	1440	8196	4864	1439	7917	15
50	7953	5065	2116	9098	6008	2840	9589	6249	2816	9285	10
55	9381	6489	3534	240510	7414	4239	290981	7633	4193	340653	5
60	190809	7912	4951	1922	8819	5637	2372	9017	5568	2020	0
Cos.	79°	78°	77°	76°	75°	74°	73°	72°	71°	70°	
Sin.	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	
0	342020	358368	374607	390731	406737	422618	438371	453990	469472	484810	60
5	3387	9725	5955	2070	8065	3936	9678	5286	470755	6081	55
10	4752	361082	7302	3407	9392	5253	440984	6580	2038	7352	50
15	6117	2438	8649	4744	410719	6569	2289	7874	3320	8621	45
20	7481	3793	9994	6080	2045	7884	3593	9166	4600	9890	40
25	8845	5148	381339	7415	3369	9198	4896	460458	5880	491157	35
30	350207	6501	2683	8749	4693	430511	6198	1749	7159	2424	30
35	1569	7854	4027	400082	6016	1823	7499	3038	8436	3689	25
40	2931	9206	5369	1415	7338	3135	8799	4327	9713	4953	20
45	4291	370557	6711	2747	8660	4445	450098	5615	480989	6217	15
50	5651	1908	8052	4078	9980	5755	1397	6901	2263	7479	10
55	7010	3258	9392	5408	421300	7063	2694	8187	3537	8740	5
60	8368	4607	390731	6737	2618	8371	3990	9472	4810	500000	0
Cos.	69°	68°	67°	66°	65°	64°	63°	62°	61°	60°	
Sin.	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	
0	500000	515038	529919	544639	559193	573576	587785	601815	615661	629320	60
5	1259	6284	531152	5858	560398	4767	8961	2976	6807	630450	55
10	2517	7529	2384	7076	1602	5957	590136	4136	7951	1578	50
15	3774	8773	3615	8293	2805	7145	1310	5294	9094	2705	45
20	5030	520016	4844	9509	4007	8332	2482	6451	620235	3831	40
25	6285	1258	6072	550724	5207	9518	3653	7607	1376	4955	35
30	7538	2499	7300	1937	6406	580703	4823	8761	2515	6078	30
35	8791	3738	8526	3149	7604	1886	5991	9915	3652	7200	25
40	510043	4977	9751	4360	8801	3069	7159	611067	4789	8320	20
45	1293	6214	540974	5570	9997	4250	8325	2217	5923	9439	15
50	2543	7450	2197	6779	571191	5429	9489	3367	7057	640557	10
55	3791	8685	3419	7987	2384	6608	600653	4515	8189	1673	5
60	5038	9919	4639	9193	3576	7785	1815	5661	9320	2788	0
′	59°	58°	57°	56°	55°	54°	53°	52°	51°	50°	′

NATURAL COSINES.

41°	42°	43°	44°	45°	46°	47°	48°	49°	'
656059	669131	681998	694658	707107	719340	731354	743145	754710	60
7156	670211	3061	5704	8134	720349	2345	4117	5663	55
8252	1289	4123	6748	9161	1357	3334	5088	6615	50
9346	2367	5183	7790	710185	2364	4323	6057	7565	45
660439	3443	6242	8832	1209	3369	5309	7025	8514	40
1530	4517	7299	9871	2230	4372	6294	7991	9461	35
2620	5590	8355	700909	3250	5374	7277	8956	760406	30
3709	6662	9409	1946	4269	6375	8259	9919	1350	25
4796	7732	690462	2981	5286	7374	9239	750880	2292	20
5882	8801	1513	4015	6302	8371	740218	1840	3232	15
6966	9868	2563	5047	7316	9367	1195	2798	4171	10
8049	680934	3611	6078	8329	730361	2171	3755	5109	5
9131	1998	4658	7107	9340	1354	3145	4710	6044	0
48°	47°	46°	45°	44°	43°	42°	41°	40°	

51°	52°	53°	54°	55°	56°	57°	58°	59°	'
777146	788011	798636	809017	819152	829038	838671	848048	857167	60
8060	8905	9510	9871	9985	9850	9462	8818	7915	55
8973	9798	800383	810723	820817	830661	840251	9586	8662	50
9884	790690	1254	1574	1647	1470	1039	850352	9406	45
780794	1579	2123	2423	2475	2277	1825	1117	860149	40
1702	2467	2991	3270	3302	3082	2609	1879	0890	35
2608	3353	3857	4116	4126	3886	3391	2640	1629	30
3513	4238	4721	4959	4949	4688	4172	3399	2366	25
4416	5121	5584	5801	5770	5488	4951	4156	3102	20
5317	6002	6445	6642	6590	6286	5728	4912	3836	15
6217	6882	7304	7480	7407	7083	6503	5665	4567	10
7114	7759	8161	8317	8223	7878	7277	6417	5297	5
8011	8636	9017	9152	9038	8671	8048	7167	6025	0
38°	37°	36°	35°	34°	33°	32°	31°	30°	

61°	62°	63°	64°	65°	66°	67°	68°	69°	'
874620	882948	891007	898794	906308	913545	920505	927184	933580	60
5324	3629	1666	9431	6922	4136	1072	7728	4101	55
6026	4309	2323	900065	7533	4725	1638	8270	4619	50
6727	4988	2979	0698	8143	5311	2201	8810	5135	45
7425	5664	3633	1329	8751	5896	2762	9348	5650	40
8122	6338	4284	1958	9357	6479	3322	9884	6162	35
8817	7011	4934	2585	9961	7060	3880	930418	6672	30
9510	7681	5582	3210	910563	7639	4435	0950	7181	25
880201	8350	6229	3834	1164	8216	4989	1480	7687	20
0891	9017	6873	4455	1762	8791	5541	2008	8191	15
1578	9682	7515	5075	2358	9364	6090	2534	8694	10
2264	890345	8156	5692	2953	9936	6638	3058	9194	5
2948	1007	8794	6308	3545	920505	7184	3580	9693	0
28°	27°	26°	25°	24°	23°	22°	21°	20°	

71°	72°	73°	74°	75°	76°	77°	78°	79°	'
945519	951057	956305	961262	965926	970296	974370	978148	981627	60
5991	1505	6729	1662	6301	0647	4696	8449	1904	55
6462	1951	7151	2059	6675	0995	5020	8748	2178	50
6930	2396	7571	2455	7046	1342	5342	9045	2450	45
7397	2838	7990	2849	7415	1687	5662	9341	2721	40
7861	3279	8406	3241	7782	2029	5980	9634	2989	35
8324	3717	8820	3630	8148	2370	6296	9925	3255	30
8784	4153	9232	4018	8511	2708	6610	980214	3519	25
9243	4588	9642	4404	8872	3045	6921	0500	3781	20
9699	5020	960050	4787	9231	3379	7231	0785	4041	15
950154	5450	0456	5169	9588	3712	7539	1068	4298	10
0606	5879	0860	5548	9943	4042	7844	1349	4554	5
1057	6305	1262	5926	970296	4370	8148	1627	4808	0
18°	17°	16°	15°	14°	13°	12°	11°	10°	'

NATURAL COSINES

′	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°
0	000000	017452	034899	052336	069756	087156	104528	121869	139173	[illegible]
5	1454	8907	6353	3788	071207	8605	5975	3313	140613	[illegible]
10	2909	020361	7806	5241	2658	090053	7421	4756	2053	[illegible]
15	4363	1815	9260	6693	4108	1502	8867	6199	3493	[illegible]
20	5818	3269	040713	8145	5559	2950	110313	7642	4932	[illegible]
25	7272	4723	2166	9597	7009	4398	1758	9084	6371	[illegible]
30	8727	6177	3619	061049	8459	5846	3203	130526	7809	[illegible]
35	010181	7631	5072	2500	9909	7293	4648	1968	9248	[illegible]
40	1635	9085	6525	3952	081359	8741	6093	3410	150686	[illegible]
45	3090	030539	7978	5403	2808	100188	7537	4851	2123	[illegible]
50	4544	1992	9431	6854	4258	1635	8982	6292	3561	[illegible]
55	5998	3446	050883	8306	5707	3082	120426	7733	4998	[illegible]
60	7452	4899	2336	9756	7156	4528	1869	9173	6434	[illegible]
Cos.	89°	88°	87°	86°	85°	84°	83°	82°	81°	[illegible]

Sin.	10°	11°	12°	13°	14°	15°	16°	17°	18°	[illegible]
0	173648	190809	207912	224951	241922	258819	275637	292372	309017	[illegible]
5	5080	2237	9334	6368	3333	260224	7035	3762	310446	[illegible]
10	6512	3664	210756	7784	4743	1628	8432	5152	1782	[illegible]
15	7944	5090	2178	9200	6153	3031	9829	6542	3164	[illegible]
20	9375	6517	3599	230616	7563	4434	281225	7930	4545	[illegible]
25	180805	7942	5019	2031	8972	5837	2620	9318	5925	[illegible]
30	2236	9368	6440	3445	250380	7238	4015	300706	7305	[illegible]
35	3665	200793	7859	4859	1788	8640	5410	2093	8684	[illegible]
40	5095	2218	9279	6273	3195	270040	6803	3479	320062	[illegible]
45	6524	3642	220697	7686	4602	1440	8196	4864	1439	[illegible]
50	7953	5065	2116	9098	6008	2840	9589	6249	2816	[illegible]
55	9381	6489	3534	240510	7414	4239	290981	7633	4193	[illegible]
60	190809	7912	4951	1922	8819	5637	2372	9017	5568	[illegible]
Cos.	79°	78°	77°	76°	75°	74°	73°	72°	71°	[illegible]

Sin.	20°	21°	22°	23°	24°	25°	26°	27°	28°
0	342020	358368	374607	390731	406737	422618	438371	453990	469472
5	3387	9725	5955	2070	8065	3936	9678	5286	470755
10	4752	361082	7302	3407	9392	5253	440984	6580	2038
15	6117	2438	8649	4744	410719	6569	2289	7874	3320
20	7481	3793	9994	6080	2045	7884	3593	9166	4600
25	8845	5148	381339	7415	3369	9198	4896	460458	5880
30	350207	6501	2683	8749	4693	430511	6198	1749	7159
35	1569	7854	4027	400082	6016	1823	7499	3038	8436
40	2931	9206	5369	1415	7338	3135	8799	4327	9713
45	4291	370557	6711	2747	8660	4445	450098	5615	480989
50	5651	1908	8052	4078	9980	5755	1397	6901	2263
55	7010	3258	9392	5408	421300	7063	2694	8187	3537
60	8368	4607	390731	6737	2618	8371	3990	9472	4810
Cos.	69°	68°	67°	66°	65°	64°	63°	62°	61°

Sin.	30°	31°	32°	33°	34°	35°	36°	37°	38°
0	500000	515038	529919	544639	559193	573576	587785	601815	615661
5	1259	6284	531152	5858	560398	4767	8961	2976	6807
10	2517	7529	2384	7076	1602	5957	590136	4136	7951
15	3774	8773	3615	8293	2805	7145	1310	5294	9094
20	5030	520016	4844	9509	4007	8332	2482	6451	620235
25	6285	1258	6072	550724	5207	9518	3653	7607	1376
30	7538	2499	7300	1937	6406	580703	4823	8761	2515
35	8791	3738	8526	3149	7604	1886	5991	9915	3652
40	510043	4977	9751	4360	8801	3069	7159	611067	4789
45	1293	6214	540974	5570	9997	4250	8325	2217	5923
50	2543	7450	2197	6779	571191	5429	9489	3367	7057
55	3791	8685	3419	7987	2384	6608	600653	4515	8189
60	5038	9919	4639	9193	3576	7785	1815	5661	9320
′	59°	58°	57°	56°	55°	54°	53°	52°	51°

41°	42°	43°	44°	45°	46°	47°	48°	49°	'
656059	669131	681998	694658	707107	719340	731354	743145	754710	60
7156	670211	3061	5704	8134	720349	2345	4117	5663	55
8252	1289	4123	6748	9161	1357	3334	5088	6615	50
9346	2367	5183	7790	710185	2364	4323	6057	7565	45
660439	3443	6242	8832	1209	3369	5309	7025	8514	40
1530	4517	7299	9871	2230	4372	6294	7991	9461	35
2620	5590	8355	700909	3250	5374	7277	8956	760406	30
3709	6662	9409	1946	4269	6375	8259	9919	1350	25
4796	7732	690462	2981	5286	7374	9239	750880	2292	20
5882	8801	1513	4015	6302	8371	740218	1840	3232	15
6966	9868	2563	5047	7316	9367	1195	2798	4171	10
8049	680934	3611	6078	8329	730361	2171	3755	5109	5
9131	1998	4658	7107	9340	1354	3145	4710	6044	0
48°	47°	46°	45°	44°	43°	42°	41°	40°	

51°	52°	53°	54°	55°	56°	57°	58°	59°	'
777146	788011	798636	809017	819152	829038	838671	848048	857167	60
8060	8905	9510	9871	9985	9850	9462	8818	7915	55
8973	9798	800383	810723	820817	830661	840251	9586	8662	50
9884	790690	1254	1574	1647	1470	1039	850352	9406	45
780794	1579	2123	2423	2475	2277	1825	1117	860149	40
1702	2467	2991	3270	3302	3082	2609	1879	0890	35
2608	3353	3857	4116	4126	3886	3391	2640	1629	30
3513	4238	4721	4959	4949	4688	4172	3399	2366	25
4416	5121	5584	5801	5770	5488	4951	4156	3102	20
5317	6002	6445	6642	6590	6286	5728	4912	3836	15
6217	6882	7304	7480	7407	7083	6503	5665	4567	10
7114	7759	8161	8317	8223	7878	7277	6417	5297	5
8011	8636	9017	9152	9038	8671	8048	7167	6025	0
38°	37°	36°	35°	34°	33°	32°	31°	30°	

61°	62°	63°	64°	65°	66°	67°	68°	69°	'
874620	882948	891007	898794	906308	913545	920505	927184	933580	60
5324	3629	1666	9431	6922	4136	1072	7728	4101	55
6026	4309	2323	900065	7533	4725	1638	8270	4619	50
6727	4988	2979	0698	8143	5311	2201	8810	5135	45
7425	5664	3633	1329	8751	5896	2762	9348	5650	40
8122	6338	4284	1958	9357	6479	3322	9884	6162	35
8817	7011	4934	2585	9961	7060	3880	930418	6672	30
9510	7681	5582	3210	910563	7639	4435	0950	7181	25
880201	8350	6229	3834	1164	8216	4989	1480	7687	20
0891	9017	6873	4455	1762	8791	5541	2008	8191	15
1578	9682	7515	5075	2358	9364	6090	2534	8694	10
2264	890345	8156	5692	2953	9936	6638	3058	9194	5
2948	1007	8794	6308	3545	920505	7184	3580	9693	0
28°	27°	26°	25°	24°	23°	22°	21°	20°	

71°	72°	73°	74°	75°	76°	77°	78°	79°	'
945519	951057	956305	961262	965926	970296	974370	978148	981627	60
5991	1505	6729	1662	6301	0647	4696	8449	1904	55
6462	1951	7151	2059	6675	0995	5020	8748	2178	50
6930	2396	7571	2455	7046	1342	5342	9045	2450	45
7397	2838	7990	2849	7415	1687	5662	9341	2721	40
7861	3279	8406	3241	7782	2029	5980	9634	2989	35
8324	3717	8820	3630	8148	2370	6296	9925	3255	30
8784	4153	9232	4018	8511	2708	6610	980214	3519	25
9243	4588	9642	4404	8872	3045	6921	0500	3781	20
9699	5020	960050	4787	9231	3379	7231	0785	4041	15
950154	5450	0456	5169	9588	3712	7539	1068	4298	10
0606	5879	0860	5548	9943	4042	7844	1349	4554	5
1057	6305	1262	5926	970296	4370	8148	1627	4808	0
18°	17°	16°	15°	14°	13°	12°	11°	10°	'

60	7688	0268	2546	4522	6195	7564	8630
Cos.	9°	8°	7°	6°	5°	4°	3°

NATURAL COSINES.

NATURAL TANGENTS.

′	0°	1°	2°	3°	4°	5°	6°
0	000000	017455	034921	052408	069927	087489	105104
5	1454	8910	6377	3866	071389	8954	6575
10	2909	020365	7834	5325	2851	090421	8046
15	4363	1820	9290	6784	4313	1887	9518
20	5818	3275	040747	8243	5775	3354	110990
25	7272	4731	2204	9703	7238	4821	2463
30	8727	6186	3661	061163	8702	6289	3936
35	010181	7641	5118	2623	080165	7757	5409
40	1636	9097	6576	4083	1629	9226	6883
45	3091	030553	8033	5543	3094	100695	8358
50	4545	2009	9491	7004	4558	2164	9833
55	6000	3465	050949	8465	6023	3634	121309
60	7455	4921	2408	9927	7489	5104	2785
Cot.	89°	88°	87°	86°	85°	84°	83°

Tan.	10°	11°	12°	13°	14°	15°	16°
0	176327	194380	212557	230868	249328	267949	286745
5	7827	5890	4077	2401	250873	9509	8320
10	9328	7401	5599	3934	2420	271069	9896
15	180830	8912	7121	5469	3968	2631	291473
20	2332	200425	8645	7004	5517	4194	3052
25	3835	1938	220169	8541	7066	5759	4632
30	5339	3452	1695	240079	8618	7325	6214
35	6844	4967	3221	1618	260170	8892	7796
40	8350	6483	4749	3157	1723	280460	9380
45	9856	8000	6277	4698	3278	2029	300966
50	191363	9518	7806	6241	4834	3600	2553
55	2871	211037	9337	7784	6391	5172	4141
60	4380	2557	230868	9328	7949	6745	5731
Cot.	79°	78°	77°	76°	75°	74°	73°

A TABLE

OF THE

AREAS OF CIRCULAR SEGMENTS.

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.001	.000042	.051	.015119	.101	.041476	.151	.074589	.201	.112624
.002	.000119	.052	.015561	.102	.042080	.152	.075306	.202	.113426
.003	.000219	.053	.016007	.103	.042687	.153	.076026	.203	.114230
.004	.000337	.054	.016457	.104	.043296	.154	.076747	.204	.115035
.005	.000470	.055	.016911	.105	.043908	.155	.077469	.205	.115842
.006	.000618	.056	.017369	.106	.044522	.156	.078194	.206	.116650
.007	.000779	.057	.017831	.107	.045139	.157	.078921	.207	.117460
.008	.000951	.058	.018296	.108	.045759	.158	.079649	.208	.118271
.009	.001135	.059	.018766	.109	.046381	.159	.080380	.209	.119084
.010	.001329	.060	.019239	.110	.047005	.160	.081112	.210	.119897
.011	.001533	.061	.019716	.111	.047632	.161	.081846	.211	.120712
.012	.001746	.062	.020196	.112	.048262	.162	.082582	.212	.121529
.013	.001969	.063	.020691	.113	.048894	.163	.083320	.213	.122347
.014	.002199	.064	.021168	.114	.049528	.164	.084059	.214	.123167
.015	.002438	.065	.021659	.115	.050165	.165	.084801	.215	.123988
.016	.002685	.066	.022154	.116	.050804	.166	.085544	.216	.124810
.017	.002940	.067	.022652	.117	.051446	.167	.086289	.217	.125634
.018	.003202	.068	.023154	.118	.052090	.168	.087036	.218	.126459
.019	.003471	.069	.023659	.119	.052736	.169	.087785	.219	.127285
.020	.003748	.070	.024168	.120	.053385	.170	.088535	.220	.128113
.021	.004031	.071	.024680	.121	.054036	.171	.089287	.221	.128942
.022	.004322	.072	.025195	.122	.054689	.172	.090041	.222	.129773
.023	.004618	.073	.025714	.123	.055345	.173	.090797	.223	.130605
.024	.004921	.074	.026236	.124	.056003	.174	.091554	.224	.131438
.025	.005230	.075	.026761	.125	.056663	.175	.092313	.225	.132272
.026	.005546	.076	.027289	.126	.057326	.176	.093074	.226	.133108
.027	.005867	.077	.027821	.127	.057991	.177	.093836	.227	.133945
.028	.006194	.078	.028356	.128	.058658	.178	.094601	.228	.134784
.029	.006527	.079	.028894	.129	.059327	.179	.095366	.229	.135624
.030	.006865	.080	.029435	.130	.059999	.180	.096134	.230	.136465
.031	.007209	.081	.029979	.131	.060672	.181	.096904	.231	.137307
.032	.007558	.082	.030526	.132	.061348	.182	.097674	.232	.138150
.033	.007913	.083	.031076	.133	.062026	.183	.098447	.233	.138995
.034	.008273	.084	.031629	.134	.062707	.184	.099221	.234	.139841
.035	.008638	.085	.032186	.135	.063389	.185	.099997	.235	.140688
.036	.009008	.086	.032745	.136	.064074	.186	.100774	.236	.141537
.037	.009383	.087	.033307	.137	.064760	.187	.101553	.237	.142387
.038	.009763	.088	.033872	.138	.065449	.188	.102334	.238	.143238
.039	.010148	.089	.034441	.139	.066140	.189	.103116	.239	.144091
.040	.010537	.090	.035011	.140	.066833	.190	.103900	.240	.144944
.041	.010931	.091	.035585	.141	.067528	.191	.104685	.241	.145799
.042	.011330	.092	.036162	.142	.068225	.192	.105472	.242	.146655
.043	.011734	.093	.036741	.143	.068924	.193	.106261	.243	.147512
.044	.012142	.094	.037323	.144	.069625	.194	.107051	.244	.148371
.045	.012554	.095	.037909	.145	.070328	.195	.107842	.245	.149230
.046	.012971	.096	.038497	.146	.071033	.196	.108636	.246	.150091
.047	.013392	.097	.039087	.147	.071741	.197	.109431	.247	.150953
.048	.013818	.098	.039680	.148	.072450	.198	.110226	.248	.151816
.049	.014247	.099	.040276	.149	.073161	.199	.111025	.249	.152680
.050	.014681	.100	.040875	.150	.073874	.200	.111823	.250	.153546

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4
60	962611	050304	144507	246037	355852	4
Cot.	27°	26°	25°	24°	23°	
Tan.	70°	71°	72°	73°	74°	
0	2.747477	2.904211	3.077684	3.270853	3.487414	3.7
5	759961	917991	092983	287949	506656	7
10	772545	931889	108421	305209	526094	7
15	785231	945905	123999	322636	545733	7
20	798029	960042	139719	340233	565575	8
25	810913	974302	155584	358001	585624	8
30	823913	988685	171595	375943	605884	8
35	837020	3.003194	187754	394063	626357	8
40	850235	017830	204064	412363	647047	9
45	863560	032595	220526	430845	667958	9
50	876997	047492	237144	449512	689093	9
55	890547	062520	253918	468368	710456	9
60	904211	077684	270853	487414	732051	4.0
Cot.	19°	18°	17°	16°	15°	1
Tan.	78°	79°	80°	81°	82°	8
0	4.704630	5.144554	5.671282	6.313752	7.115370	8.1
5	738508	184804	719917	373736	191246	2
10	772857	225665	769369	434843	268726	3
15	807685	267152	819657	497104	347861	4
20	843005	309279	870804	560554	428706	5
25	878825	352063	922832	625226	511318	6
30	915157	395517	975764	691156	595754	7
35	952013	439659	6.029625	758383	682077	8
40	989403	484505	084438	826944	770351	9.0
45	5.027340	530072	140230	896880	860642	1
50	065835	576379	197028	968234	953022	2
55	104902	623442	254859	7.041048	8.047565	3
60	144554	671282	313752	115370	144346	5
Cot.	11°	10°	9°	8°	7°	
Tan.	86°	Diff.	87°	Diff.	88°	D